

# Studies on Santilli's Locally Anisotropic and Inhomogeneous Isogeometries: I. Isobundles and Generalized Isofinsler Gravity

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## Abstract

We generalize the geometry of Santilli's locally anisotropic and inhomogeneous isospaces to the geometry of vector isobundles provided with nonlinear and distinguished isoconnections and isometric structures. We present, apparently for the first time, the isotopies of Lagrange, Finsler and Kaluza–Klein spaces. We also continue the study of the interior, locally anisotropic and inhomogeneous gravitation by extending the isoriemannian space's constructions and presenting a geometric background for the theory of isofield interactions in generalized isolagrange and isofinsler spaces.

In press at *Algebras, Groups and Geometries*, Vol. 14, 1997  
1991 Math. Subj. Class. 51N30, 51p05, 53-XX



# 1 Introduction

A number of physical problems connected with the general interior dynamics of deformable particles while moving within inhomogeneous and anisotropic physical media result in a study of the most general known systems which are nonlinear in coordinates  $x$  and their derivatives  $\dot{x}, \ddot{x}, \dots$ , on wave functions and  $\psi$  and their derivatives  $\partial\psi, \partial\partial\psi, \dots$ . Such systems are also nonlocal because of possible integral dependencies on all of the proceeding quantities and noncanonical with violation of integrability conditions for the existence of a Lagrangian or a Hamiltonian [19].

The mathematical methods for a quantitative treatment of the latter nonlinear, nonlocal and nonhamiltonian systems have been identified by Santilli in a series of contributions beginning the late 1970's [20, 21, 23] under the name of isotopies, and include axiom preserving liftings of fields of numbers, vector and metric spaces, differential and integral calculus, algebras and geometries. These studies were then continued by a number of authors (see ref. [4] for a comprehensive literature up to 1985, and monographs [10, 12, 19, 21, 22, 26] for subsequent literature).

This paper is devoted to a study of Santilli's isospaces and isogeometries over isofields treated via the isodifferential calculus according to their latest formulation [22] (we extend this calculus for isospaces provided with nonlinear isoconnection structure). We shall also use Kadeisvili's notion of isocontinuity [10] and the novel Santilli–Tsagas–Sourlas isodifferential topology [23, 26].

After reviewing the basic elements for completeness as well as for notational convenience, we shall extend Santilli's foundations of the isosymplectic geometry [23] to isobundles and related aspects (by applying, in an isotopic manner, the methods summarized in Miron and Anastasiei [14] and Yano and Ishihara [34] monographs). We shall study, apparently for the first time, the isotopies of Lagrange, Finsler and Kaluza–Klein geometries. We shall then apply the results to further studies of the isogravitational theories (for isoriemannian spaces firstly considered by Santilli [23]) on vector isobundle provided with compatible nonlinear and distinguished isoconnections and isometric structures. Such isogeometrical models of isofield interaction isotherories are in general nonlinear, nonlocal and nonhamiltonian and contain a very large class of local anisotropies and inhomogeneities induced by



four fundamental isostructures: the partition of unity, nonlinear isoconnection, distinguished isoconnections and isometric.

The novel geometric profile emerging from all the above studies is rather remarkable inasmuch as the first class of all isotopies herein considered (called Kadeisvili's Class I [10]) preserves the abstract axioms of conventional formulations, yet permits a clear broadening of their applicability, and actually result to be "directly universal" [23] for a number of possible well behaved nonlinear, nonlocal and nonhamiltonian systems. In turn, this permits a number of geometric unification such as that of all possible metrics (on isospaces with trivial nonlinear isoconnection structure) of a given dimension into Santilli's isoeuclidean metric, the unification of exterior and interior gravitational problems despite their sizable structural differences and other unification.

The view adopted in this work is that a general field theory should incorporate all possible anisotropic, inhomogeneous and stochastic manifestations of classical and quantum interactions and, in consequence, corresponding modifications of basic principles and mathematical methods have to be introduced in order to formulate physical theories. There are established three approaches for modeling field interactions and spaces anisotropies. The first one is to deal with a usual locally isotropic physical theory and to consider anisotropies as a consequence of the anisotropic structure of sources in field equations (for instance, a number of cosmological models are proposed in the framework of the Einstein theory with the energy-momentum generated by anisotropic matter, as a general reference see [16]). The second approach to anisotropies originates from the Finsler geometry [9, 8, 18, 13] and its generalizations [3, 14, 2, 5, 28, 27] with a general imbedding into Kaluza-Klein (super) gravity and string theories [29, 30, 31], and speculates a generic anisotropy of the space-time structure and of fundamental field of interactions. The Santilli's approach is more radical by proposing a generalization of Lie theory and introducing isofields, isodualities and related mathematical structures. Roughly speaking, by using corresponding partitions of the unit we can model possible metric anisotropies as in Finsler or generalized Lagrange geometry but the problem is also to take into account classes of anisotropies generated by nonlinear and distinguished connections.

This is a first paper from a series of works devoted to the formulation of the theory of inhomogeneous, locally anisotropic and higher order



anisotropic isofield interactions (here we note that the term "higher order" is used as a general one for higher order tangent bundles [34], or higher order extensions of vector superbundles [30, 31], in a number of lines alternative to jet bundles [25, 24], and only under corresponding constraints one obtains the geometry of higher order Lagrangians [15]). The main purpose of this paper is to present a synthesis of the Santilli isothory and the approach on modeling locally anisotropic geometries and physical models on bundle spaces provided with nonlinear connection and distinguished connection and metric structures [14, 34]. The isotopic variants of generalized Lagrange and Finsler geometry will be analyzed. Basic geometric constructions such as nonlinear isoconnections in vector isobundles, the isotopic curvatures and torsions of distinguished isoconnections and their structure equations and invariant values will be defined. A model of locally anisotropic and inhomogeneous gravitational isothory will be constructed.

Section 2 is devoted to basic notations and definitions on Santilli and coauthors isothory. We introduce the bundle isospaces in Section 3 where some necessary properties of Lie–Santilli isoalgebras and isogroups and corresponding isotopic extensions of manifolds are applied in order to define fiber isospaces and consider their such (being very important for modeling of isofield interactions) classes of principal isobundles and vector isobundles. The isogeometry of nonlinear isoconnections in vector isobundles is studied in Section 4. Isotopic distinguishing of geometric objects, the isocurvatures and isotorsions of nonlinear and distinguished isoconnections, the structure equations and invariant conditions are defined in Section 5. The next Section 6 is devoted to the isotopic extensions of generalized Lagrange and Finsler geometries. The isofield equations of locally anisotropic and inhomogeneous interactions will be analyzed in Section 7. The outlook and conclusions are contained in Section 8.

## 2 Basic Notions on Isotopies

In this section we shall mainly recall some necessary fundamental notions and refer to works [23, 10] for details and references on Lie–Santilli isothory.



## 2.1 Isotopies of the unit and isospaces

The **isotheory** is based on the concept of fundamental **isotopy** which is the lifting  $I \rightarrow \hat{I}$  of the  $n$ -dimensional unit  $I = \text{diag}(1, 1, \dots, 1)$  of the Lie's theory into an  $n \times n$ -dimensional matrix

$$\hat{I} = (I_j^i) = \hat{I}(t, x, \dot{x}, \ddot{x}, \psi, \psi^+, \partial\psi, \partial\psi^+, \partial\partial\psi, \partial\partial\psi^+, \dots)$$

called the isounit. For simplicity, we consider that maps  $I \rightarrow \hat{I}$  are of necessary Kadeisvili Class I (II), the Class III being considered as the union of the first two, i. e. they are sufficiently smooth, bounded, nowhere degenerate, Hermitian and positive (negative) definite, characterizing isotopies (isodualities).

One demands a compatible lifting of all associative products  $AB$  of some generic quantities  $A$  and  $B$  into the isoproduct  $A * B$  satisfying the properties:

$$AB \Rightarrow A * B = \hat{A}\hat{T}B, \quad IA = AI \equiv A \rightarrow \hat{I} * A = A * \hat{I} \equiv A,$$

$$A(BC) = (AB)C \rightarrow A * (B * C) = (A * B) * C,$$

where the fixed and invertible matrix  $\hat{T}$  is called the isotopic element.

To follow our outline, a conventional field  $F(a, +, \times)$ , for instance of real, complex or quaternion numbers, with elements  $a$ , conventional sum  $+$  and product  $a \times b \doteq ab$ , must be lifted into the so-called isofield  $\hat{F}(\hat{a}, +, *)$ , satisfying properties

$$F(a, +, *) \rightarrow \hat{F}(\hat{a}, +, *), \quad \hat{a} = a\hat{I}$$

$$\hat{a} * \hat{b} = \hat{a}\hat{T}\hat{b} = (ab)\hat{I}, \quad \hat{I} = \hat{T}^{-1}$$

with elements  $\hat{a}$  called isonumbers,  $+$  and  $*$  are conventional sum and isoproduct preserving the axioms of the former field  $F(a, +, \times)$ . All operations in  $F$  are generalized for  $\hat{F}$ , for instance we have isosquares  $\hat{a}^2 = \hat{a} * \hat{a} = \hat{A}\hat{T}\hat{a} = a^2\hat{I}$ , isoquotient  $\hat{a}/\hat{b} = (a/b)\hat{I}$ , isosquare roots  $\hat{a}^{1/2} = a^{1/2}\hat{I}, \dots$ ;  $\hat{a} * A \equiv aA$ . We note that in the literature one uses two types of denotation for isotopic product  $*$  or  $\hat{\times}$  (in our work we shall consider  $*$   $\equiv$   $\hat{\times}$ ).

Let us consider, for example, the main lines of the isotopies of a  $n$ -dimensional Euclidean space  $E^n(x, g, \mathcal{R})$ , where  $\mathcal{R}(n, +, \times)$  is the real number



field, provided with a local coordinate chart  $x = \{x^k\}, k = 1, 2, \dots, n$ , and  $n$ -dimensional metric  $\rho = (\rho_{ij}) = \text{diag}(1, 1, \dots, 1)$ . The scalar product of two vectors  $x, y \in E^n$  is defined as

$$(x - y)^2 = (x^i - y^i) \rho_{ij} (x^j - y^j) \in \mathcal{R}(n, +, \times)$$

were the Einstein summation rule on repeated indices is assumed hereon.

The **Santilli's isoeuclidean** spaces  $\hat{E}(\hat{x}, \hat{\rho}, \hat{R})$  of Class III are introduced as  $n$ -dimensional metric spaces defined over an isoreal isofield  $\hat{R}(\hat{n}, +, \hat{\times})$  with an  $n \times n$ -dimensional real-valued and symmetrical isounit  $\hat{I} = \hat{I}^t$  of the same class, equipped with the "isometric"

$$\hat{\rho}(t, x, v, a, \mu, \tau, \dots) = (\hat{\rho}_{ij}) = \hat{T}(t, x, v, a, \mu, \tau, \dots) \times \rho = \hat{\rho}^t,$$

where  $\hat{I} = \hat{T}^{-1} = \hat{I}^t$ .

A local coordinate cart on  $\hat{E}(\hat{x}, \hat{\rho}, \hat{R})$  can be defined in contravariant

$$\hat{x} = \{\hat{x}^k = x^k\} = \{x^k \times \hat{I}_k^k\}$$

or covariant form

$$\hat{x}_k = \hat{\rho}_{kl} \hat{x}^l = \hat{T}_k^r \rho_{ri} x^i \times \hat{I},$$

where  $x^k, x_k \in \hat{E}$ . The square of "isoeuclidean distance" between two points  $\hat{x}, \hat{y} \in \hat{E}$  is defined as

$$(\hat{x} - \hat{y})^2 = [(\hat{x}^i - \hat{y}^i) \times \hat{\rho}_{ij} \times (\hat{x}^j - \hat{y}^j)] \times \hat{I} \in \hat{R}$$

and the isomultiplication is given by

$$\hat{x}^2 = \hat{x}^k \hat{\times} \hat{x}_k = (x^k \times \hat{I}) \times \hat{T} \times (x_k \times \hat{I}) = (x^k \times x_k) \times \hat{I} = n \times \hat{I}.$$

Whenever confusion does not arise isospaces can be practically treated via the conventional coordinates  $x^k$  rather than the isotopic ones  $\hat{x}^k = x^k \times \hat{I}$ . The symbols  $x, v, a, \dots$  will be used for conventional spaces while symbols  $\hat{x}, \hat{v}, \hat{a}, \dots$  will be used for isospaces; the letter  $\hat{\rho}(x, v, a, \dots)$  refers to the projection of the isometric  $\hat{\rho}$  in the original space.

We note that an isofield of Class III, explicitly denoted as  $\hat{F}_{III}(\hat{a}, +, \hat{\times})$  is a union of two disjoint isofields, one of Class I,  $\hat{F}_I(\hat{a}, +, \hat{\times})$ , in which



the isounit is positive definite, and one of Class II,  $\widehat{F}_{II}(\widehat{a}, +, \widehat{\times})$ , in which the isounit is negative-definite. The Class II of isofields is usually written as  $\widehat{F}^d(\widehat{a}^d, +, \widehat{\times}^d)$  and called isodual fields with isodual unit  $\widehat{I}^d = -\widehat{I} < 0$ , isodual isonumbers  $\widehat{a}^d = a \times \widehat{I}^d = -\widehat{a}$ , isodual isoproduct  $\widehat{\times}^d = \times \widehat{I}^d \times = -\widehat{\times}$ , etc. For simplicity, in our further considerations we shall use the general terms isofields, isonumbers even for isodual fields, isodual numbers and so on if this will not give rise to ambiguities.

## 2.2 Isocontinuity and isotopology

The isonorm of an isofield of Class III is defined as

$$\uparrow \widehat{a} \uparrow = |a| \times \widehat{I}$$

where  $|a|$  is the conventional norm. Having defined a function  $\widehat{f}(\widehat{x})$  on isospace  $\widehat{E}(\widehat{x}, \widehat{\delta}, \widehat{R})$  over isofield  $\widehat{R}(\widehat{n}, +, \widehat{\times})$  one introduces (see details and references in [10]) the isomodulus

$$\uparrow \widehat{f}(\widehat{x}) \uparrow = |\widehat{f}(\widehat{x})| \times \widehat{I}$$

where  $|\widehat{f}(\widehat{x})|$  is the conventional modulus.

One says that an infinite sequence of isofunctions of Class I  $\widehat{f}_1, \widehat{f}_2, \dots$  is "strongly isoconvergent" to the isofunction  $\widehat{f}$  of the same class if

$$\lim_{k \rightarrow \infty} \uparrow \widehat{f}_k - \widehat{f} \uparrow = \widehat{0}.$$

The Cauchy isocondition is expressed as

$$\uparrow \widehat{f}_m - \widehat{f}_n \uparrow < \widehat{\rho} = \rho \times \widehat{I}$$

where  $\delta$  is real and  $m$  and  $n$  are greater than a suitably chosen  $N(\rho)$ . Now the isotopic variants of continuity, limits, series, etc, can be easily constructed in a traditional manner.

The notion of  $n$ -dimensional isomanifold was studied by Tsagas and Sourlas (we refer the reader for details in [26]). Their constructions are



based on idea that every isounit of Class III can always be diagonalized into the form

$$\hat{I} = \text{diag} (B_1, B_2, \dots, B_n), B_k(x, \dots) \neq 0, k = 1, 2, \dots, n.$$

In result of this one defines an isotopology  $\hat{\tau}$  on  $\hat{R}^n$  which coincides everywhere with the conventional topology  $\tau$  on  $R^n$  except at the isounit  $\hat{I}$ . In particular,  $\hat{\tau}$  is everywhere local-differential, except at  $\hat{I}$  which can incorporate integral terms. The above structure is called the Tsagas-Sourlas isotopology or an integro-differential topology. Finally, in this subsection, we note that Prof. Tsagas and Sourlas used a conventional topology on isomanifolds. The isotopology was first introduced by Prof. Santilli in ref. [23].

### 2.3 Isodifferential and isointegral calculus

Now we are able to introduce isotopies of the ordinary differential calculus, i.e. the isodifferential calculus (for short).

The **isodifferentials** of Class I of the contravariant and covariant coordinates  $\hat{x}^k = x^{\hat{k}}$  and  $\hat{x}_k = x_{\hat{k}}$  on an isoeuclidean space  $\hat{E}$  of the same class is given by

$$\hat{d}\hat{x}^k = \hat{I}^k_i(x, \dots) dx^i, \quad \hat{d}\hat{x}_k = \hat{T}_k^i(x, \dots) dx_i \quad (2.1)$$

where  $\hat{d}\hat{x}^k$  and  $\hat{d}\hat{x}_k$  are defined on  $\hat{E}$  while the  $\hat{I}^k_i dx^i$  and  $\hat{T}_k^i dx_i$  are the projections on the conventional Euclidean space.

For a sufficiently smooth isofunction  $\hat{f}(\hat{x})$  on a closed domain  $\hat{U}(\hat{x}^k)$  covered by contravariant isocoordinates  $\hat{x}^k$  we can define the partial isoderivatives  $\hat{\partial}_k = \frac{\hat{\partial}}{\hat{\partial}\hat{x}^k}$  at a point  $\hat{x}_{(0)}^k \in \hat{U}(\hat{x}^k)$  by considering the limit

$$\hat{f}'(\hat{x}_{(0)}^k) = \hat{\partial}_k \hat{f}(\hat{x})|_{\hat{x}_{(0)}^k} = \frac{\hat{\partial}\hat{f}(\hat{x})}{\hat{\partial}\hat{x}^k}|_{\hat{x}_{(0)}^k} = \hat{T}_k^i \frac{\partial f(x)}{\partial x^i}|_{\hat{x}_{(0)}^k} = \quad (2.2)$$

$$\lim_{\hat{d}\hat{x}^k \rightarrow \hat{\partial}^k} \frac{\hat{f}(\hat{x}_{(0)}^k + \hat{d}\hat{x}^k) - \hat{f}(\hat{x}_{(0)}^k)}{\hat{d}\hat{x}^k}$$

where  $\hat{\partial}\hat{f}(\hat{x})/\hat{\partial}\hat{x}^k$  is computed on  $\hat{E}$  and  $\hat{T}_k^i \partial f(x)/\partial x^i$  is the projection in  $E$ .



In a similar manner we can define the **partial isoderivatives**  $\widehat{\partial}^k = \frac{\widehat{\partial}}{\widehat{\partial}x_k}$  with respect to a covariant variable  $\widehat{x}_k$  :

$$\begin{aligned} \widehat{f}'(\widehat{x}_{k(0)}) &= \widehat{\partial}^k \widehat{f}(\widehat{x})|_{\widehat{x}_{k(0)}} = \frac{\widehat{\partial} \widehat{f}(\widehat{x})}{\widehat{\partial} \widehat{x}_k}|_{\widehat{x}_{k(0)}} = \widehat{T}_k^i \frac{\partial f(x)}{\partial x^i}|_{\widehat{x}_{k(0)}} = \\ & \lim_{\widehat{dx}_k \rightarrow \widehat{0}_k} \frac{\widehat{f}(\widehat{x}_{k(0)} + \widehat{dx}_k) - \widehat{f}(\widehat{x}_{k(0)})}{\widehat{dx}_k}. \end{aligned} \quad (2.3)$$

The isodifferentials of an isofunction of contravariant or covariant coordinates,  $\widehat{x}^k$  or  $\widehat{x}_k$ , are defined according the formulas

$$\widehat{d}\widehat{f}(\widehat{x})|_{contrav} = \widehat{\partial}_k \widehat{f} \widehat{dx}^k = \widehat{T}_k^i \frac{\partial f(x)}{\partial x^i} \widehat{I}_j^k dx^j = \frac{\partial f(x)}{\partial x^k} dx^k = \frac{\partial f(x)}{\partial x^i} \widehat{T}_j^i dx^j$$

and

$$\widehat{d}\widehat{f}(\widehat{x})|_{covar} = \widehat{\partial}^k \widehat{f} \widehat{dx}_k = \widehat{I}_i^k \frac{\partial f(x)}{\partial x_i} \widehat{T}_k^j dx_j = \frac{\partial f(x)}{\partial x_k} dx_k = \frac{\partial f}{\partial x_j} \widehat{I}_j^i dx_i.$$

The second order isoderivatives there are introduced by iteration of the notion of isoderivative:

$$\begin{aligned} \widehat{\partial}_{\widehat{i}\widehat{j}}^2 \widehat{f}(\widehat{x}) &= \frac{\widehat{\partial}^2 \widehat{f}(\widehat{x})}{\widehat{\partial} \widehat{x}^i \widehat{\partial} \widehat{x}^j} = \widehat{T}_i^i \widehat{T}_j^j \frac{\partial^2 f(x)}{\partial x^i \partial x^j}, \\ \widehat{\partial}^{2\widehat{i}\widehat{j}} \widehat{f}(\widehat{x}) &= \frac{\widehat{\partial}^2 \widehat{f}(\widehat{x})}{\widehat{\partial} \widehat{x}_i \widehat{\partial} \widehat{x}_j} = \widehat{I}_i^i \widehat{I}_j^j \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \\ \widehat{\partial}_i^2 \widehat{f}(\widehat{x}) &= \frac{\widehat{\partial}^2 \widehat{f}(\widehat{x})}{\widehat{\partial} \widehat{x}^i \widehat{\partial} \widehat{x}_{\widehat{j}}} = \widehat{T}_i^i \widehat{I}_j^j \frac{\partial^2 f(x)}{\partial x^i \partial x_j}. \end{aligned}$$

The Laplace isooperator on Euclidean space  $\widehat{E}(\widehat{x}, \widehat{\delta}, \widehat{R})$  is given by

$$\widehat{\Delta} = \widehat{\partial}_k \widehat{\partial}^k = \widehat{\partial}^i \rho_{ij} \widehat{\partial}^j = \widehat{I}_k^i \partial^k \rho_{ij} \partial^j \quad (2.4)$$

where there are also used usual partial derivatives  $\partial^j = \partial/\partial x_j$  and  $\partial_k = \partial/\partial x^k$ .



The isodual isodifferential calculus is characterized by the following isodual differentials and isodual isoderivatives

$$\widehat{d}^{(d)}\widehat{x}^{(d)k} = \widehat{I}^{(d)k}_i d\widehat{x}^{(d)i} \equiv \widehat{d}x^k, \quad \widehat{\partial}^{(d)}/\widehat{\partial}\widehat{x}^{(d)i} \widehat{T}_k^{i(d)} \partial/\partial\widehat{x}^{i(d)} \equiv \widehat{T}_k^i \partial/\partial\widehat{x}^i.$$

The formula (2.4) is different from the expression for the Laplace operator

$$\Delta = \widehat{\rho}^{-1/2} \partial_i \widehat{\rho}^{1/2} \widehat{\rho}^{ij} \partial_j$$

even though the Euclidean isometric  $\widehat{\rho}(x, v, a, \dots)$  is more general than the Riemannian metric  $g(x)$ . For partial isoderivations one follows the next properties:

$$\frac{\widehat{\partial}\widehat{x}^i}{\widehat{\partial}\widehat{x}^j} = \delta^i_j, \quad \frac{\widehat{\partial}\widehat{x}_i}{\widehat{\partial}\widehat{x}_j} = \delta_i^j, \quad \frac{\widehat{\partial}\widehat{x}_i}{\widehat{\partial}\widehat{x}^j} = \widehat{T}_i^j, \quad \frac{\widehat{\partial}\widehat{x}^i}{\widehat{\partial}\widehat{x}_j} = \widehat{I}^i_j.$$

Here we remark that isointegration (the inverse to isodifferential) is defined [23] as to satisfy conditions

$$\int^\wedge \widehat{d}\widehat{x} = \int \widehat{T}\widehat{I}dx = \int dx = x,$$

where  $\int^\wedge = \int \widehat{T}$ .

## 2.4 Santilli's isoriemannian isospaces

Let consider  $\mathcal{R} = \mathcal{R}(x, g, R)$  a (pseudo) Riemannian space over the reals  $R(n, +, \times)$  with local coordinates  $x = \{x^\mu\}$  and nonwere singular, symmetrical and real-valued metric  $g(x) = (g_{\mu\nu}) = g^t$  and the tangent flat space  $M(x, \eta, R)$  provided with flat real metric  $\eta$  (for a corresponding signature and dimension we can consider  $M$  as the well known Minkowski space). The metric properties of the Riemannian spaces are defined by scalar square of a tangent vector  $x$ ,

$$x^2 = x^\mu g_{\mu\nu}(x) x^\nu \in R$$

or, in infinitesimal form by the line element

$$ds^2 = dx^\mu g_{\mu\nu}(x) dx^\nu$$

and related formalism of covariant derivation (see for instance [16]).



The isotopies of the Riemannian spaces and geometry, were first studied and applied by [23] and are called Santilli's isoriemannian spaces and geometry. In this section we consider isoriemannian spaces equipped with the Santilli–Tsagas–Sourlas isotopology [23, 26] in a similar manner as we have done in the previous subsection for isoeuclidean spaces but with respect to a general, non flat, isometric. A isoriemannian space  $\widehat{\mathcal{R}} = \widehat{\mathcal{R}}(\widehat{x}, \widehat{g}, \widehat{R})$ , over the isoreals  $\widehat{R} = \widehat{R}(\widehat{n}, +, \widehat{x})$  with common isounits  $\widehat{I} = (\widehat{I}^\mu_\nu) = \widehat{T}^{-1}$ , is provided with local isocoordinates  $\widehat{x} = \{\widehat{x}^\mu\} = \{x^\mu\}$  and isometric  $\widehat{g}(x, v, a, \mu, \tau, \dots) = \widehat{T}(x, v, \mu, \tau, \dots) g(x)$ , where  $\widehat{T} = (\widehat{T}^\mu_\nu)$  is nowhere singular, real valued and symmetrical matrix of Class I with  $C^\infty$  elements. The corresponding isoline and infinitesimal elements are written as

$$\widehat{x}^2 = [\widehat{x}^\mu \widehat{g}_{\mu\nu}(x, v, a, \mu, \tau, \dots) \widehat{x}^\nu] \times \widehat{I} \in \widehat{R}$$

with infinitesimal version

$$d\widehat{s}^2 = (\widehat{d}\widehat{x}^\mu \widehat{g}_{\mu\nu}(x) \widehat{d}\widehat{x}^\nu) \times \widehat{I} \in \widehat{R}.$$

The **covariant isodifferential calculus** has been introduced in ref. [23] via the expression

$$\widehat{D}\widehat{X}^\beta = \widehat{d}\widehat{X}^\beta + \widehat{\Gamma}^\beta_{\alpha\gamma} \widehat{X}^\alpha \widehat{d}\widehat{x}^\gamma$$

with corresponding covariant isoderivative

$$\widehat{X}^\beta_{\uparrow\mu} = \widehat{\partial}_\mu \widehat{X}^\beta + \widehat{\Gamma}^\beta_{\alpha\mu} \widehat{X}^\alpha$$

with the **isocristoffel symbols** written as

$$\{\widehat{\alpha\beta\gamma}\} = \frac{1}{2} (\widehat{\partial}_\alpha \widehat{g}_{\beta\gamma} + \widehat{\partial}_\gamma \widehat{g}_{\alpha\beta} - \widehat{\partial}_\beta \widehat{g}_{\alpha\gamma}) = \{\gamma\widehat{\beta\alpha}\}, \quad (2.5)$$

$$\widehat{\Gamma}^\beta_{\alpha\gamma} = \widehat{g}^{\beta\rho} \{\widehat{\alpha\rho\gamma}\} = \widehat{\Gamma}^\beta_{\alpha\gamma},$$

where  $\widehat{g}^{\beta\rho}$  is inverse to  $\widehat{g}_{\alpha\beta}$ .

The crucial difference between Riemannian spaces and isospaces is obvious if the corresponding auto-parallel equations

$$\frac{Dx_\beta}{Ds} = \frac{dv_\beta}{ds} + \{\alpha\beta\gamma\}(x) \frac{dx^\alpha}{ds} \frac{dx^\gamma}{ds} = 0 \quad (2.6)$$



and auto-isoparallel equations

$$\frac{\widehat{D}\widehat{x}_\beta}{\widehat{D}\widehat{s}} = \frac{\widehat{d}v_\beta}{\widehat{d}\widehat{s}} + \{\widehat{\alpha\beta\gamma}\}(\widehat{x}, \widehat{v}, \widehat{a}, \dots) \frac{\widehat{d}\widehat{x}^\alpha}{\widehat{d}\widehat{s}} \frac{\widehat{d}\widehat{x}^\gamma}{\widehat{d}\widehat{s}} = 0 \quad (2.7)$$

where  $\widehat{v} = \widehat{d}\widehat{x}/\widehat{d}\widehat{s} = \widehat{I}_S \times dx/ds$ ,  $\widehat{s}$  is the proper isotime and  $\widehat{I}_S$  is the related one-dimensional isounit, can be identified by observing that equations (2.6) are at most quadratic in the velocities while the isotopic equations (2.7) are arbitrary nonlinear in the velocities and another possible variables and parameters  $(\widehat{a}, \dots)$ .

By using coefficients  $\widehat{\Gamma}_{\alpha\gamma}^\beta$  we introduce the next isotopic values [23]:  
the **isocurvature tensor**

$$\widehat{R}_{\alpha\gamma}{}^\beta{}_\delta = \widehat{\partial}_\delta \widehat{\Gamma}_{\alpha\gamma}^\beta - \widehat{\partial}_\gamma \widehat{\Gamma}_{\alpha\delta}^\beta + \widehat{\Gamma}_{\varepsilon\delta}^\beta \widehat{\Gamma}_{\alpha\gamma}^\varepsilon - \widehat{\Gamma}_{\varepsilon\gamma}^\beta \widehat{\Gamma}_{\alpha\delta}^\varepsilon; \quad (2.8)$$

the **isoricci tensor**  $\widehat{R}_{\alpha\gamma} = \widehat{R}_{\alpha\gamma}{}^\beta{}_\beta$ ;  
the **isocurvature scalar**  $\widehat{R} = \widehat{g}^{\alpha\gamma} \widehat{R}_{\alpha\gamma}$ ;  
the **isoeinstein tensor**

$$\widehat{G}_{\mu\nu} = \widehat{R}_{\mu\nu} - \frac{1}{2} \widehat{g}_{\mu\nu} \widehat{R} \quad (2.9)$$

and the **istopic isoscalar**

$$\widehat{\Theta} = \widehat{g}^{\alpha\beta} \widehat{g}^{\gamma\delta} (\{\widehat{\rho\alpha\delta}\} \widehat{\Gamma}_{\gamma\beta}^\rho - \{\widehat{\rho\alpha\beta}\} \widehat{\Gamma}_{\gamma\delta}^\rho) \quad (2.10)$$

(the later is a new object for the Riemannian isometry).

The isotopic lifting of the Einstein equations (see the history, details and references in [23]) is written as

$$\widehat{R}^{\alpha\beta} - \frac{1}{2} \widehat{g}^{\alpha\beta} (\widehat{R} + \widehat{\Theta}) = \widehat{t}^{\alpha\beta} - \widehat{\tau}^{\alpha\beta}, \quad (2.11)$$

where  $\widehat{t}^{\alpha\beta}$  is a **source isotensor** and  $\widehat{\tau}^{\alpha\beta}$  is the **stress-energy isotensor** and there is satisfied the Freud isidentity [23]

$$\widehat{G}^\alpha{}_\beta - \frac{1}{2} \delta^\alpha{}_\beta \widehat{\Theta} = \widehat{U}^\alpha{}_\beta + \widehat{\partial}_\rho \widehat{V}^{\alpha\rho}{}_\beta, \quad (2.12)$$

$$\widehat{U}^\alpha{}_\beta = -\frac{1}{2} \frac{\widehat{\partial}\widehat{\Theta}}{\widehat{\partial}\widehat{g}^{\gamma\delta}{}_{\uparrow\alpha}} \widehat{g}^{\gamma\delta}{}_{\uparrow\beta},$$



$$\widehat{V}^{\alpha\rho}_{\beta} = \frac{1}{2}[\widehat{g}^{\gamma\delta}(\delta^{\alpha}_{\beta}\widehat{\Gamma}^{\rho}_{\alpha\delta} - \delta^{\alpha}_{\delta}\widehat{\Gamma}^{\rho}_{\alpha\beta}) + \widehat{g}^{\rho\gamma}\widehat{\Gamma}^{\alpha}_{\beta\gamma} - \widehat{g}^{\alpha\gamma}\widehat{\Gamma}^{\rho}_{\beta\gamma} + (\delta^{\rho}_{\beta}\widehat{g}^{\alpha\gamma} - \delta^{\alpha}_{\beta}\widehat{g}^{\rho\gamma})\widehat{\Gamma}^{\rho}_{\gamma\rho}].$$

Finally, we remark that for antiautomorphic maps of isoduality we have to modify correspondingly the above presented formulas holding true for Riemannian isodual spaces  $\widehat{\mathcal{R}}^{(d)} = \widehat{\mathcal{R}}^{(d)}(\widehat{x}^{(d)}, \widehat{g}^{(d)}, \widehat{R}^{(d)})$ , over the isodual reals  $\widehat{R}^{(d)} = \widehat{R}^{(d)}(\widehat{n}^{(d)}, +, \widehat{\times}^{(d)})$  with curvature, Ricci, Einstein and so on isodual tensors. For simplicity we omit such details in this work.

### 3 Isobundle Spaces

This section serves the twofold purpose of establishing of abstract index denotations and starting the geometric backgrounds of isotopic locally anisotropic extensions of the isoriemannian spaces which are used in the next sections of the work.

#### 3.1 Lie–Santilli isoalgebras and isogroups

The Lie–Santilli isothory is based on a generalization of the very notion of numbers and fields. If the Lie’s theory is centrally dependent on the basic  $n$ -dimensional unit  $I = \text{diag}(1, 1, \dots, 1)$  in, for instance, enveloping algebras, Lie algebras, Lie groups, representation theory, and so on, the Santilli’s main idea is the reformulation of the entire conventional theory with respect to the most general possible, integro–differential isounit. In this subsection we introduce some necessary definitions and formulas on Lie–Santilli isoalgebra and isogroups following [10] where details, developments and basic references on Santilli original result are contained. A Lie–Santilli algebra is defined as a finite–dimensional isospaces  $\widehat{L}$  over the isofield  $\widehat{F}$  of isoreal or isocomplex numbers with isotopic element  $T$  and isounit  $\widehat{I} = T^{-1}$ . In brief one uses the term isoalgebra (when there is not confusion with isotopies of non–Lie algebras) which is defined by isolinear isocommutators of type  $[A, \wedge B] \in \widehat{L}$  satisfying the conditions:

$$\begin{aligned} [A, \wedge B] &= -[B, \wedge A], \\ [A, \wedge [B, \wedge C]] + [B, \wedge [C, \wedge A]] + [C, \wedge [A, \wedge B]] &= 0, \\ [A * B, \wedge C] &= A * [B, \wedge C] + [A, \wedge C] * B \end{aligned}$$



for all  $A, B, C \in \hat{L}$ . The structure functions  $\hat{C}$  of the Lie–Santilli algebras are introduced according the relations

$$[X_i, \hat{\cdot} X_j] = X_i * X_j - X_j * X_i = \\ X_i T(x, \dots) X_j - X_j T(x, \dots) X_i = \hat{C}_{ij}^k(x, \dot{x}, \ddot{x}, \dots) * X_k.$$

It should be noted that, in fact, the basis  $e_k$ , ( $k = 1, 2, \dots, N$ ) of a Lie algebra  $L$  is not changed under isotopy except the renormalization factors  $\hat{e}_k$ : the isocommutation rules of the isotopies  $\hat{L}$  are

$$[\hat{e}_i, \hat{e}_j] = \hat{e}_i T \hat{e}_j - \hat{e}_j T \hat{e}_i = \hat{C}_{ij}^k(x, \dot{x}, \ddot{x}, \dots) \hat{e}_k$$

where  $\hat{C} = CT$ .

An isomatrix  $\hat{M}$  is and ordinary matrix whose elements are isoscalars. All operations among isomatrices are therefore isotopic.

The isotrace of a isomatrix  $A$  is introduced by using the unity  $\hat{I}$ :

$$\widehat{Tr} A = (Tr A) \hat{I} \in \hat{F}$$

where  $Tr A$  is the usual trace. One holds properties

$$\widehat{Tr}(A * B) = (\widehat{Tr} A) * (\widehat{Tr} B)$$

and

$$\widehat{Tr} A = \widehat{Tr} (BAB^{-1}).$$

The Killing isoform is determined by the isoscalar product

$$(A, \hat{\cdot} B) = \widehat{Tr} [(\widehat{Ad} X) * (\widehat{Ad} B)]$$

where the isolinear maps are introduced as  $\widehat{ad} A(B) = [A, \hat{\cdot} B], \forall A, B \in \hat{L}$ . Let  $e_k, k = 1, 2, \dots, N$  be the basis of a Lie algebra with an isomorphic map  $e_k \rightarrow \hat{e}_k$  to the basis  $\hat{e}_k$  of a Lie–Santilli isoalgebra  $\hat{L}$ . We can write the elements in  $\hat{L}$  in local coordinate form. For instance, considering  $A = x^i \hat{e}_i, B = y^j \hat{e}_j$  and  $C = z^k \hat{e}_k = [A, \hat{\cdot} B]$  we have

$$C = z^k \hat{e}_k = [A, \hat{\cdot} B] = x^i y^j [\hat{e}_i, \hat{\cdot} \hat{e}_j] = x^i x^j \hat{C}_{ij}^k \hat{e}_k$$

and

$$[\widehat{Ad} A(B)]^k = [A, \hat{\cdot} B]^k = x^i x^j \hat{C}_{ij}^k.$$



In standard manner there is introduced the **isocartan tensor**

$$\hat{q}_{ij}(x, \dot{x}, \ddot{x}, \dots) = \hat{C}_{ip}^k \hat{C}_{ik}^p \in \hat{L}$$

via the definition

$$(A, \wedge B) = \hat{q}_{ij} x^i y^j.$$

Considering that  $\hat{L}$  is an isoalgebra with generators  $X_k$  and isounit  $\hat{I} = T^{-1} > 0$  the isodual Lie–Santilli algebras  $\hat{L}^d$  of  $\hat{L}$  (we note that  $\hat{L}$  and  $\hat{L}^d$  are (anti) isomorphic).

The conventional structure of the Lie theory admits a conventional isotopic lifting. Let give some examples. The general isolinear and isocomplex Lie–Santilli algebras  $\hat{gl}(n, \hat{C})$  are introduced as the vector isospaces of all  $n \times n$  isocomplex matrices over  $\hat{C}$ . For the isoreal numbers  $\hat{R}$  we shall write  $\hat{gl}(n, \hat{R})$ . By using "hats" we denote respectively the special, isocomplex, isolinear isoalgebra  $\hat{sl}(n, \hat{C})$  and the isoorthogonal algebra  $\hat{o}(n)$ .

A right Lie–Santilli isogroup  $\hat{Gr}$  on an isospace  $\hat{S}(x, \hat{F})$  over an isofield  $\hat{F}, \hat{I} = T^{-1}$  (in brief isotransformation group or isogroup) is introduced in standard form but with respect to isonumbers and isofields as a group which maps each element  $x \in \hat{S}(x, \hat{F})$  into a new element  $x' \in \hat{S}(x, \hat{F})$  via the isotransformations  $x' = \hat{U} * x = \hat{U}Tx$ , where  $T$  is fixed such that

1. The map  $(U, x) \rightarrow \hat{U} * x$  of  $\hat{Gr} \times \hat{S}(x, \hat{F})$  onto  $\hat{S}(x, \hat{F})$  is isodifferentiable;
2.  $\hat{I} * \hat{U} = \hat{U} * \hat{I} = \hat{U}, \forall \hat{U} \in \hat{Gr}$ ;
3.  $\hat{U}_1 * (\hat{U}_2 * x) = (\hat{U}_1 * \hat{U}_2) * x, \forall x \in \hat{S}(x, \hat{F})$  and  $\hat{U}_1, \hat{U}_2 \in \hat{Gr}$ .

We can define accordingly a left isotransformation group.

## 3.2 Fiber isobundles

Prof. Santilli identified the foundations of the isosymplectic geometry in the work [23]. In this section we present, apparently for the first time, the isotopies of fibre bundles and related topics.

The notion of locally trivial fiber isobundle naturally generalizes that of the isomanifold. The fiber isobundles will be used to get some results in isogeometry as well as to build geometrical models for physical isotheories. In general the proofs, being corresponding reformulation in isotopic manner



of standard results, will be omitted. The reader is referred to some well-known books containing the theory of fibre bundles and the mathematical foundations of the Lie–Santilli isothory.

Let  $\widehat{Gr}$  be a Lie–Santilli isogroup which acts isodifferentiably and effectively on a isomanifold  $\widehat{V}$ , i.e. every element  $\widehat{g} \in \widehat{Gr}$  defines an isotopic diffeomorphism  $L_{\widehat{Gr}} : \widehat{V} \rightarrow \widehat{V}$ .

As a rule, all isomanifolds are assumed to be isocontinuous, finite dimensional and having the isotopic variants of the conditions to be Hausdorff, paracompact and isoconnected; all isomaps are isocontinuous.

A locally trivial **fibre isobundle** is defined by the data  $(\widehat{E}, \widehat{p}, \widehat{M}, \widehat{V}, \widehat{Gr})$ , where  $\widehat{M}$  (the base isospace) and  $\widehat{E}$  (the total isospace) are isomanifolds,  $\widehat{E}, \widehat{p} : \widehat{E} \rightarrow \widehat{M}$  is a surjective isomap and the following conditions are satisfied:

1/ the isomanifold  $\widehat{M}$  can be covered by a set  $\mathcal{E}$  of open isotopic sets  $\widehat{U}, \widehat{W}, \dots$  such that for every open set  $\widehat{U}$  there exist a bijective isomap  $\widehat{\varphi}_{\widehat{U}} : \widehat{p}^{-1}(\widehat{U}) \rightarrow \widehat{U} \times \widehat{V}$  so that  $\widehat{p}(\varphi_{\widehat{U}}^{-1}(\widehat{x}, \widehat{y})) = \widehat{x}, \forall \widehat{x} \in \widehat{U}, \forall \widehat{y} \in \widehat{V}$ ;

2/ if  $\widehat{x} \in \widehat{U} \cap \widehat{W} \neq \emptyset$ , then  $\widehat{\varphi}_{\widehat{W},x} \circ \widehat{\varphi}_{\widehat{U},x}^{-1} : \widehat{V} \rightarrow \widehat{V}$  is an isotopic diffeomorphism  $L_{\widehat{Gr}}$  with  $\widehat{g}r \in \widehat{Gr}$  where  $\widehat{\varphi}_{\widehat{U},x}$  denotes the restriction of  $\widehat{\varphi}_{\widehat{U}}$  to  $\widehat{p}^{-1}(\widehat{x})$  and  $\widehat{U}, \widehat{W} \in \mathcal{E}$ ;

3/ the isomap  $q_{\widehat{U}\widehat{V}} : \widehat{U} \cap \widehat{V} \rightarrow \widehat{Gr}$  defined by structural isofunctions  $q_{\widehat{U}\widehat{V}}(\widehat{x}) = \widehat{\varphi}_{\widehat{W},x} \circ \widehat{\varphi}_{\widehat{U},x}^{-1}$  is isocontinuous.

Let  $In_U$  and  $In_V$  be sets of indices and denote by  $(\widehat{U}_\alpha, \widehat{\varphi}_\alpha)_{\alpha \in In_U}$  and  $(\widehat{V}_\beta, \widehat{\psi}_\beta)_{\beta \in In_V}$  be correspondingly isocontinuous atlases on  $\widehat{U}_\alpha$  and  $\widehat{V}_\beta$ . One obtains an isotopic topology on  $\widehat{E}$  for which the bijections  $\widehat{\varphi}_{\widehat{U}}, \widehat{\varphi}_{\widehat{W},x} \dots$  become isotopic homeomorphisms. Denoting respectively by  $n$  and  $m$  the dimensions of the isomanifolds  $\widehat{M}$  and  $\widehat{V}$  we can define the isotopic maps

$$\phi_{\alpha\beta} : \varpi_{\alpha\beta} \rightarrow \widehat{R}^{n+m}, \phi_{\alpha\beta} = (\widehat{\varphi}_\alpha \times \widehat{\psi}_\beta) \circ \varphi_{\widehat{U}}^{\alpha\beta}$$

where  $\varphi_{\widehat{U}}^{\alpha\beta}$  is the restriction to the isomap  $\widehat{\varphi}_{\widehat{U}}$  to  $\varpi_{\alpha\beta}$ . Than the set  $(\varpi_{\alpha\beta}, \phi_{\alpha\beta})_{(\alpha,\beta) \in In_U \times In_V}$  is a isocontinuous atlas on  $\widehat{E}$ .

A locally trivial **principal isobundle**  $(\widehat{P}, \widehat{\pi}, \widehat{M}, \widehat{Gr})$  is a fibre isobundle  $(\widehat{E}, \widehat{p}, \widehat{M}, \widehat{V}, \widehat{Gr})$  for which the type fibre coincides with the structural



group,  $\widehat{V} = \widehat{G}r$  and the action of  $\widehat{G}r$  on  $\widehat{G}r$  is given by the left isotransform  $\widehat{L}_q(a) = qa, \forall q, a \in \widehat{G}r$ .

The structural functions of the principal isobundle  $(\widehat{P}, \widehat{\pi}, \widehat{M}, \widehat{G}r)$  are

$$q_{\widehat{U}\widehat{W}} : \widehat{U}, \widehat{W} \rightarrow \widehat{G}r, \quad q_{\widehat{U}\widehat{W}}(\widehat{\pi}(u)) = \widehat{\varphi}_{\widehat{W}}(u) \circ \widehat{\varphi}_{\widehat{U}}^{-1}(u), u \in \widehat{\pi}^{-1}(\widehat{U} \cap \widehat{W}).$$

**A morphism of principal isobundles**  $(\widehat{P}, \widehat{\pi}, \widehat{M}, \widehat{G}r)$  and  $(\widehat{P}', \widehat{\pi}', \widehat{M}', \widehat{G}r')$  is a pair  $(\widehat{f}, \widehat{f}')$  of isomaps for which the following conditions hold:

- 1/  $\widehat{f} : \widehat{P} \rightarrow \widehat{P}'$  is a isocontinuous isomap,
- 2/  $\widehat{f} : \widehat{G}r \rightarrow \widehat{G}r'$  is an isotopic morphism of Lie–Santilli isogroups.
- 3/  $\widehat{f}(\widehat{u}\widehat{q}) = \widehat{f}(\widehat{u})\widehat{f}'(\widehat{q}), \widehat{u} \in \widehat{P}, \widehat{q} \in \widehat{G}r$ .

We can define isotopic isomorphisms, automorphisms and subbundles in a usual manner but with respect to isonumbers, isofields, isogroups and isomanifold when corresponding isotopic transforms and maps provide the isotopic properties.

**A isotopic subbundle**  $(\widehat{P}, \widehat{\pi}, \widehat{M}, \widehat{G}r)$  of the principal isobundle  $(\widehat{P}', \widehat{\pi}', \widehat{M}', \widehat{G}r')$  is called a reduction of the structural isogroup  $\widehat{G}r'$  to  $\widehat{G}r$ .

An isotopic frame (isoframe) in a point  $\widehat{x} \in \widehat{M}$  is a set of  $n$  linearly independent isovectors tangent to  $\widehat{M}$  in  $\widehat{x}$ . The set  $\widehat{L}(\widehat{M})$  of all isoframes in all points of  $\widehat{M}$  can be naturally provided (as in the non isotopic case, see, for instance, [11]) with an isomanifold structure. The principal isobundle  $(\widehat{L}(\widehat{M}), \widehat{\pi}, \widehat{M}, \widehat{Gl}(n, \widehat{R}))$  of isoframes on  $\widehat{M}$ , denoted in brief also by  $\widehat{L}(\widehat{M})$ , has  $\widehat{L}(\widehat{M})$  as the total space and the general linear isogroup  $\widehat{Gl}(n, \widehat{R})$  as the structural isogroup.

Having introduced the isobundle  $\widehat{L}(\widehat{M})$  we can give define an **isotopic  $G$ -structure** on a isomanifold  $\widehat{M}$  is a subbundle  $(\widehat{P}, \widehat{\pi}, \widehat{M}, \widehat{G}r)$  of the principal isobundle  $\widehat{L}(\widehat{M})$  being an isotopic reduction of the structural isogroup  $\widehat{Gl}(n, \widehat{R})$  to a isotopic subgroup  $\widehat{G}r$  of it.

A very important class of bundle spaces used for modeling of locally anisotropic interactions is that of vector bundles. We present here the necessary isotopic generalizations.



An locally trivial **isovector bundle** (equivalently, **vector isobundle**, **v-isobundle**)  $(\widehat{E}, \widehat{p}, \widehat{M}, \widehat{V}, \widehat{G}r)$  is defined as a corresponding fibre isobundle if  $\widehat{V}$  is a linear isospace and  $\widehat{G}r$  is the Lie–Santilli isogroup of isotopic automorphisms of  $\widehat{V}$ .

For  $\widehat{V} = \widehat{R}^m$  and  $\widehat{G}r = \widehat{G}l(m, \widehat{R})$  the v-isobundle  $(\widehat{E}, \widehat{p}, \widehat{M}, \widehat{R}^m, \widehat{G}l(m, \widehat{R}))$  is denoted shortly as  $\widehat{\xi} = (\widehat{E}, \widehat{p}, \widehat{M})$ . Here we also note that the transformations of the isocoordinates  $(\widehat{x}^k, \widehat{y}^a) \rightarrow (\widehat{x}^{k'}, \widehat{y}^{a'})$  on  $\widehat{\xi}$  are of the form

$$\widehat{x}^{k'} = \widehat{x}^{k'}(\widehat{x}^1, \dots, \widehat{x}^n), \quad \text{rank} \left( \frac{\partial \widehat{x}^{k'}}{\partial \widehat{x}^k} \right) = n$$

$$\widehat{y}^{a'} = \widehat{Y}_a^{a'}(x) y^a, \quad \widehat{Y}_a^{a'}(x) \in \widehat{G}l(m, \widehat{R}).$$

A local isocoordinate parametrization of  $\widehat{\xi}$  naturally defines an isocoordinate basis

$$\frac{\partial}{\partial \widehat{u}^\alpha} = \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a} \right), \quad (3.1)$$

in brief we shall write  $\widehat{\partial}_\alpha = (\widehat{\partial}_i, \widehat{\partial}_a)$ , and the reciprocal to (3.1) coordinate basis

$$d\widehat{u}^\alpha = (d\widehat{x}^i, d\widehat{y}^a),$$

or, in brief,  $\widehat{d}^\alpha = (\widehat{d}^i, \widehat{d}^a)$ , which is uniquely defined from the equations

$$\widehat{d}^\alpha \circ \widehat{\partial}_\beta = \delta_\beta^\alpha,$$

where  $\delta_\beta^\alpha$  is the Kronecher symbol and by "o" we denote the inner (scalar) product in the isotangent isobundle  $\widehat{T}\widehat{\xi}$  (see the definition of isodifferentials and partial isoderivations in (2.1)–(2.3)). Here we note that the tangent isobundle (in brief t-isobundle) of a isomanifold  $\widehat{M}$ , denoted as  $\widehat{TM} = \cup_{x \in \widehat{M}} \widehat{T}_x \widehat{M}$ , where  $\widehat{T}_x \widehat{M}$  is tangent isospaces of tangent isovectors in the point  $\widehat{x} \in \widehat{M}$ , is defined as a v-isobundle  $\widehat{E} = \widehat{TM}$ . By  $\widehat{TM}^*$  we define the dual (not confusing with isotopic dual) of the t-isobundle  $\widehat{TM}$ . We note that for  $\widehat{TM}$  and  $\widehat{TM}^*$  isobundles the fibre and the base have both the same dimension and it is not necessary to distinguish always the fiber and base indices by different letters.



## 4 Nonlinear and Distinguished Isoconnections

The concept of **nonlinear connection**, in brief, N-connection, is fundamental in the geometry of locally anisotropic spaces (in brief, la-spaces, see a detailed study and basic references in [14]). Here it should be noted that we consider the term la-space in a more general context than G. Yu. Bogoslovsky [7] which uses it for a class of Finsler spaces and Finsler gravitational theories. In our papers [27, 28, 29, 31, 32, 33] the la-spaces and la-superspaces are respectively modelled on vector bundles and vector superbundles enabled with compatible nonlinear and distinguished connections and metric structures (as particular cases, on tangent bundles and superbundles, for corresponding classes of metrics and nonlinear connections, one constructs generalized Lagrange and Finsler spaces and superspaces). The geometrical objects on a la-space are called **distinguished** (see details in [14]) if they are compatible with the N-connection structure (one considers, for instance, distinguished connections and distinguished tensors, in brief, d-connections and d-tensors).

In this section we study, apparently for the first time, the isogeometry of N-connection in vector isobundle. We note that a type of generic nonlinearity is contained by definition in the structure of isospace (it can be associated to a corresponding class of nonlinear isoconnections which can be turned into linear ones under corresponding isotopic transforms). As to a general N-connection introduced as a global decompositions of a vector isobundle into horizontal and vertical isotopic subbundles (see below) it can not be isolarized if its isocurvature is nonzero.

Let consider a v-isobundle  $\widehat{\xi} = (\widehat{E}, \widehat{p}, \widehat{M})$  whose type fibre is  $\widehat{R}^m$  and  $\widehat{p}^T : \widehat{TE} \rightarrow \widehat{TM}$  is the isodifferential of the isomap  $\widehat{p}$ . The kernel of the isomap  $\widehat{p}^T$  (which is a fibre-preserving isotopic morphism of the t-isobundle  $(\widehat{TE}, \widehat{\tau}_E, \widehat{E})$  to  $\widehat{E}$  and of t-isobundle  $(\widehat{TM}, \widehat{\tau}_M, \widehat{M})$  to  $\widehat{M}$ ) defines the vertical isotopic subbundle  $(\widehat{VE}, \widehat{\tau}_V, \widehat{E})$  over  $\widehat{E}$  being an isovector subbundle of the v-isobundle  $(\widehat{TE}, \widehat{\tau}_E, \widehat{E})$ .

An isovector  $\widehat{X}_u$  tangent to  $\widehat{E}$  in a point  $\widehat{u} \in \widehat{E}$  locally defined by the



decomposition  $\widehat{X}^i \widehat{\partial}_i + \widehat{Y}^a \widehat{\partial}_a$  is locally represented by the isocoordinates

$$\widehat{X}_u = (\widehat{x}, \widehat{y}, \widehat{X}, \widehat{Y}) = (\widehat{x}^i, \widehat{y}^a, \widehat{X}^i, \widehat{Y}^a).$$

Since  $\widehat{p}^T(\widehat{\partial}_a) = 0$  it results that  $\widehat{p}^T(\widehat{x}, \widehat{y}, \widehat{X}, \widehat{Y}) = (\widehat{x}, \widehat{X})$ . We also consider the isotopic imbedding map  $\widehat{i} : \widehat{V}E \rightarrow \widehat{T}E$  and the isobundle of inverse image  $\widehat{p}^* \widehat{T}\widehat{M}$  of the  $\widehat{p} : \widehat{E} \rightarrow \widehat{M}$  and define in result the isomap  $\widehat{p}! : \widehat{T}E \rightarrow \widehat{p}^* \widehat{T}\widehat{M}$ ,  $\widehat{p}!(\widehat{X}_u) = (\widehat{u}, \widehat{p}^T(\widehat{X}_u))$  for which one holds  $\text{Ker } \widehat{p}! = \text{Ker } \widehat{p}^T = \widehat{V}E$ .

A **nonlinear isoconnection**, (in brief, **N-isoconnection**) in the iso-vector bundle  $\widehat{\xi} = (\widehat{E}, \widehat{p}, \widehat{M})$  is defined as a splitting on the left of the exact sequence of isotopic maps

$$0 \longrightarrow \widehat{V}E \xrightarrow{\widehat{i}} \widehat{T}E \xrightarrow{\widehat{p}!} \widehat{T}E/\widehat{V}E \longrightarrow 0$$

that is an isotopic morphism of vector isobundles  $\widehat{C} : \widehat{T}E \rightarrow \widehat{V}E$  such that  $\widehat{C} \circ \widehat{i}$  is the identity on  $\widehat{V}E$ .

The kernel of the isotopic morphism  $\widehat{C}$  is a isovector subbundle of  $(\widehat{T}E, \widehat{\tau}_E, \widehat{E})$  and will be called the horizontal isotopic subbundle  $(\widehat{H}E, \widehat{\tau}_H, \widehat{E})$ .

As a consequence of the above presented definition we can consider that a N-isoconnection in v-isobundle  $\widehat{E}$  is a isotopic distribution  $\{\widehat{N} : \widehat{E}_u \rightarrow H_u \widehat{E}, T_u \widehat{E} = H_u \widehat{E} \oplus V_u \widehat{E}\}$  on  $\widehat{E}$  such that it is defined a global decomposition, as a Whitney sum, into horizontal,  $\widehat{H}E$ , and vertical,  $\widehat{V}E$ , subbundles of the tangent isobundle  $\widehat{T}E$ :

$$\widehat{T}E = \widehat{H}E \oplus \widehat{V}E. \quad (4.1)$$

Locally a N-isoconnection in  $\widehat{\xi}$  is given by its components  $\widehat{N}_i^a(\widehat{u}) = \widehat{N}_i^a(\widehat{x}, \widehat{y}) = \widehat{N}_i^a(\widehat{x}, \widehat{y})$  with respect to local isocoordinate bases (2.14) and (2.15)):

$$\widehat{\mathbf{N}} = \widehat{N}_i^a(\widehat{u}) \widehat{d}^i \otimes \widehat{\partial}_a.$$

We note that a linear isoconnection in a v-isobundle  $\widehat{\xi}$  can be considered as a particular case of a N-isoconnection when  $\widehat{N}_i^a(\widehat{x}, \widehat{y}) = \widehat{K}_{bi}^a(\widehat{x}) \widehat{y}^b$ , where isofunctions  $\widehat{K}_{ai}^b(\widehat{x})$  on the base  $\widehat{M}$  are called the isochristoffel coefficients.



To coordinate locally geometric constructions with the global splitting of isobundle defined by a N-isoconnection structure, we have to introduce a locally adapted isobasis (la-isobasis, la-isoframe):

$$\frac{\hat{\delta}}{\delta \hat{u}^\alpha} = \left( \frac{\hat{\delta}}{\delta \hat{x}^i} = \hat{\partial}_i - \hat{N}_i^a(\hat{u}) \hat{\partial}_a, \frac{\hat{\delta}}{\partial \hat{y}^a} \right), \quad (4.2)$$

or, in brief,  $\hat{\delta}_\alpha = \delta_{\hat{\alpha}} = (\hat{\delta}_i, \hat{\partial}_a)$ , and its dual la-isobasis

$$\delta \hat{u}^\alpha = (\delta \hat{x}^i = \hat{d}\hat{x}^i, \delta \hat{y}^a + \hat{N}_i^a(\hat{u}) \hat{d}\hat{x}^i), \quad (4.3)$$

or, in brief,  $\hat{\delta}^\alpha = (\hat{d}^i, \hat{\delta}^a)$ . We note that isoooperators (4.2) and (4.3) generalize correspondingly the partial isoderivations and isodifferentials (2.1)–(2.3) for the case when a N-isoconnection is defined.

The **nonholonomic isocoefficients**  $\hat{\mathbf{w}} = \{\hat{w}_{\beta\gamma}^\alpha(\hat{u})\}$  of la-isoframes are defined as

$$[\hat{\delta}_\alpha, \hat{\delta}_\beta] = \hat{\delta}_\alpha \hat{\delta}_\beta - \hat{\delta}_\beta \hat{\delta}_\alpha = \hat{w}_{\beta\gamma}^\alpha(\hat{u}) \hat{\delta}_\alpha.$$

The **algebra of tensorial distinguished isofields**  $D\hat{T}(\hat{\xi})$  (d-isofields, d-isotensors, d-tensor isofield, d-isobjects) on  $\hat{\xi}$  is introduced as the tensor algebra  $\mathcal{T} = \{\hat{T}_{qs}^{pr}\}$  of the v-isobundle  $\hat{\xi}_{(d)}$ ,  $\hat{p}_d : \hat{H}\hat{E} \oplus \hat{V}\hat{E} \rightarrow \hat{E}$ . An element  $\hat{\mathbf{t}} \in \hat{\mathcal{T}}_{qs}^{pr}$ , d-tensor isofield of type  $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ , can be written in local form as

$$\hat{\mathbf{t}} = \hat{t}_{j_1 \dots j_q b_1 \dots b_r}^{i_1 \dots i_p a_1 \dots a_r}(u) \hat{\delta}_{i_1} \otimes \dots \otimes \hat{\delta}_{i_p} \otimes \hat{\partial}_{a_1} \otimes \dots \otimes \hat{\partial}_{a_r} \otimes \hat{d}^{j_1} \otimes \dots \otimes \hat{d}^{j_q} \otimes \hat{\delta}^{b_1} \dots \otimes \hat{\delta}^{b_r}.$$

We shall respectively use denotations  $\mathcal{X}(\hat{\xi})$  (or  $\mathcal{X}(\hat{M})$ ),  $\Lambda^p(\hat{\xi})$  (or  $\Lambda^p(\hat{M})$ ) and  $\mathcal{F}(\hat{\xi})$  (or  $\mathcal{F}(\hat{M})$ ) for the isotopic module of d-vector isofields on  $\hat{\xi}$  (or  $\hat{M}$ ), the exterior algebra of p-forms on  $\hat{\xi}$  (or  $\hat{M}$ ) and the set of real functions on  $\hat{\xi}$  (or  $\hat{M}$ ).

In general, d-objects on  $\hat{\xi}$  are introduced as geometric objects with various isogroup and isocoordinate transforms coordinated with the N-connection isostructure on  $\hat{\xi}$ . For example, a d-connection  $\hat{D}$  on  $\hat{\xi}$  is defined as a isolinear connection  $\hat{D}$  on  $\hat{E}$  conserving under a parallelism the global decomposition (4.1) into horizontal and vertical subbundles of  $\hat{T}\hat{E}$ .



A N-connection in  $\widehat{\xi}$  induces a corresponding decomposition of d-isotensors into sums of horizontal and vertical parts, for example, for every d-isovector  $\widehat{X} \in \mathcal{X}(\widehat{\xi})$  and 1-form  $\widetilde{X} \in \Lambda^1(\widehat{\xi})$  we have respectively

$$\widehat{X} = \widehat{hX} + \widehat{vX} \quad \text{and} \quad \widetilde{X} = h\widetilde{X} + v\widetilde{X}.$$

In consequence, we can associate to every d-covariant isoderivation  $\widehat{D}_X = \widehat{X} \circ \widehat{D}$  two new operators of h- and v-covariant isoderivations defined respectively as

$$\widehat{D}_X^{(h)} \widehat{Y} = \widehat{D}_{hX} \widehat{Y} \quad \text{and} \quad \widehat{D}_X^{(v)} \widehat{Y} = \widehat{D}_{vX} \widehat{Y}, \quad \forall \widehat{Y} \in \mathcal{X}(\widehat{\xi})$$

for which the following conditions hold:

$$\widehat{D}_X \widehat{Y} = \widehat{D}_X^{(h)} \widehat{Y} + \widehat{D}_X^{(v)} \widehat{Y}, \quad (4.4)$$

$$\widehat{D}_X^{(h)} f = (h\widehat{X})f \quad \text{and} \quad \widehat{D}_X^{(v)} f = (v\widehat{X})f, \quad \widehat{X}, \widehat{Y} \in \mathcal{X}(\widehat{\xi}), f \in \mathcal{F}(\widehat{M}).$$

An **isometric structure**  $\widehat{\mathbf{G}}$  in the total space  $\widehat{E}$  of v-isobundle  $\widehat{\xi} = (\widehat{E}, \widehat{p}, \widehat{M})$  over a connected and paracompact base  $\widehat{M}$  is introduced as a symmetrical covariant tensor isofield of type  $(0, 2)$ ,  $\widehat{G}_{\alpha\beta}$ , being nondegenerate and of constant signature on  $\widehat{E}$ .

Nonlinear isoconnection  $\widehat{\mathbf{N}}$  and isometric  $\widehat{\mathbf{G}}$  structures on  $\widehat{\xi}$  are mutually compatible if there are satisfied the conditions:

$$\widehat{\mathbf{G}}(\widehat{\delta}_i, \widehat{\partial}_a) = 0$$

which in component form are written as

$$\widehat{G}_{ia}(\widehat{u}) - \widehat{N}_i^b(\widehat{u}) \widehat{h}_{ab}(\widehat{u}) = 0, \quad (4.5)$$

where  $\widehat{h}_{ab} = \widehat{\mathbf{G}}(\widehat{\partial}_a, \widehat{\partial}_b)$  and  $\widehat{G}_{ia} = \widehat{\mathbf{G}}(\widehat{\partial}_i, \widehat{\partial}_a)$  (the matrix  $\widehat{h}^{ab}$  is inverse to  $\widehat{h}_{ab}$ ).

In consequence one obtains the following decomposition of isotopic metric :

$$\widehat{\mathbf{G}}(\widehat{X}, \widehat{Y}) = \widehat{h}\widehat{\mathbf{G}}(\widehat{X}, \widehat{Y}) + \widehat{v}\widehat{\mathbf{G}}(\widehat{X}, \widehat{Y})$$



where the d-tensor  $\widehat{\mathbf{hG}}(\widehat{X}, \widehat{Y}) = \widehat{\mathbf{G}}(\widehat{hX}, \widehat{hY})$  is of type  $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$  and the d-isotensor  $\widehat{\mathbf{vG}}(\widehat{X}, \widehat{Y}) = \widehat{\mathbf{G}}(v\widehat{X}, v\widehat{Y})$  is of type  $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ . With respect to la-isobasis (4.3) the d-isometric is written as

$$\widehat{\mathbf{G}} = \widehat{g}_{\alpha\beta}(\widehat{u}) \widehat{\delta}^\alpha \otimes \widehat{\delta}^\beta = \widehat{g}_{ij}(\widehat{u}) \widehat{d}^i \otimes \widehat{d}^j + \widehat{h}_{ab}(\widehat{u}) \widehat{\delta}^a \otimes \widehat{\delta}^b, \quad (4.6)$$

where  $\widehat{g}_{ij} = \widehat{\mathbf{G}}(\widehat{\delta}_i, \widehat{\delta}_j)$ .

A metric isostructure of type (4.6) on  $\widehat{E}$  with components satisfying constraints (4.3)) defines an adapted to the given N-isoconnection inner (d-scalar) isoproduct on the tangent isobundle  $\widehat{TE}$ .

A d-isoconnection  $\widehat{D}_X$  is **compatible** with an isometric  $\widehat{\mathbf{G}}$  on  $\widehat{\xi}$  if

$$\widehat{D}_X \widehat{\mathbf{G}} = 0, \forall \widehat{X} \in \widehat{\mathcal{X}}(\widehat{\xi}).$$

Locally adapted components  $\widehat{\Gamma}_{\beta\gamma}^\alpha$  of a d-isoconnection  $\widehat{D}_\alpha = (\widehat{\delta}_\alpha \circ \widehat{D})$  are defined by the equations

$$\widehat{D}_\alpha \widehat{\delta}_\beta = \widehat{\Gamma}_{\alpha\beta}^\gamma \widehat{\delta}_\gamma,$$

from which one immediately follows

$$\widehat{\Gamma}_{\alpha\beta}^\gamma(\widehat{u}) = (\widehat{D}_\alpha \widehat{\delta}_\beta) \circ \widehat{\delta}^\gamma. \quad (4.7)$$

The operations of h- and v-covariant isoderivations,  $\widehat{D}_k^{(h)} = \{\widehat{L}_{jk}^i, \widehat{L}_{bk}^a\}$  and  $\widehat{D}_c^{(v)} = \{\widehat{C}_{jk}^i, \widehat{C}_{bc}^a\}$  (see (4.4)), are introduced as corresponding h- and v-parametrizations of (4.7):

$$\widehat{L}_{jk}^i = (\widehat{D}_k \widehat{\delta}_j) \circ \widehat{d}^i, \quad \widehat{L}_{bk}^a = (\widehat{D}_k \widehat{\delta}_b) \circ \widehat{\delta}^a \quad (4.8)$$

and

$$\widehat{C}_{jc}^i = (\widehat{D}_c \widehat{\delta}_j) \circ \widehat{d}^i, \quad \widehat{C}_{bc}^a = (\widehat{D}_c \widehat{\delta}_b) \circ \widehat{\delta}^a. \quad (4.9)$$

Components (4.8) and (4.9),  $\widehat{D}\widehat{\Gamma} = (\widehat{L}_{jk}^i, \widehat{L}_{bk}^a, \widehat{C}_{jc}^i, \widehat{C}_{bc}^a)$ , completely defines the local action of a d-isoconnection  $\widehat{D}$  in  $\widehat{\xi}$ . For instance, taken a d-tensor isofield of type  $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ ,

$$\widehat{\mathbf{t}} = \widehat{t}_{jb}^a \widehat{\delta}_i \otimes \widehat{\partial}_a \otimes \widehat{\partial}^j \otimes \widehat{\delta}^b,$$



and a d-vector  $\widehat{\mathbf{X}} = \widehat{X}^i \widehat{\delta}_i + \widehat{X}^a \widehat{\partial}_a$  we have

$$\widehat{D}_X \widehat{\mathbf{t}} = \widehat{D}_X^{(h)} \widehat{\mathbf{t}} + \widehat{D}_X^{(v)} \widehat{\mathbf{t}} = \left( \widehat{X}^k \widehat{t}_{jb|k}^{ia} + \widehat{X}^c \widehat{t}_{jb\perp c}^{ia} \right) \widehat{\delta}_i \otimes \widehat{\partial}_a \otimes \widehat{d}^j \otimes \widehat{\delta}^b,$$

where the **h-covariant** and **v-covariant isoderivatives** are written respectively as

$$\widehat{t}_{jb|k}^{ia} = \frac{\widehat{\delta} \widehat{t}_{jb}^{ia}}{\widehat{\delta} \widehat{x}^k} + \widehat{L}_{hk}^i \widehat{t}_{jb}^{ha} + \widehat{L}_{ck}^a \widehat{t}_{jb}^{ic} - \widehat{L}_{jk}^h \widehat{t}_{hb}^{ia} - \widehat{L}_{bk}^c \widehat{t}_{jc}^{ia}$$

and

$$\widehat{t}_{jb\perp c}^{ia} = \frac{\widehat{\partial} \widehat{t}_{jb}^{ia}}{\widehat{\partial} \widehat{y}^c} + \widehat{C}_{hc}^i \widehat{t}_{jb}^{ha} + \widehat{C}_{dc}^a \widehat{t}_{jb}^{id} - \widehat{C}_{jc}^h \widehat{t}_{hb}^{ia} - \widehat{C}_{bc}^d \widehat{t}_{jd}^{ia}.$$

For a scalar isofunction  $f \in \mathcal{F}(\widehat{\xi})$  we have

$$\widehat{D}_k^{(h)} = \frac{\widehat{\delta} f}{\widehat{\delta} \widehat{x}^k} = \frac{\widehat{\partial} f}{\widehat{\partial} \widehat{x}^k} - \widehat{N}_k^a \frac{\widehat{\partial} f}{\widehat{\partial} \widehat{y}^a} \text{ and } \widehat{D}_c^{(v)} f = \frac{\widehat{\partial} f}{\widehat{\partial} \widehat{y}^c}.$$

We emphasize that the geometry of connections in a v-isobundle  $\widehat{\xi}$  is very reach. For instance, if a triple of fundamental isogeometric objects  $(\widehat{N}_i^a(\widehat{u}), \widehat{\Gamma}_{\beta\gamma}^\alpha(\widehat{u}), \widehat{G}_{\alpha\beta}(\widehat{u}))$  is fixed on  $\widehat{\xi}$ , a multi-isoconnection structure (with corresponding rules of covariant isoderivation, which are, or not, mutually compatible and with the same, or not, induced d-scalar products in  $\widehat{TE}$ ) is defined.

Let enumerate some of isoconnections and covariant isoderivations which can present interest in investigation of locally anisotropic and homogeneous gravitational and matter field isotopic interactions:

1. Every N-isoconnection in  $\widehat{\xi}$  with coefficients  $\widehat{N}_i^a(\widehat{x}, \widehat{y})$  being isodifferentiable on y-variables induces a structure of isolinear isoconnection  $\widetilde{N}_{\beta\gamma}^\alpha$ , where  $\widetilde{N}_{bi}^a = \frac{\widehat{\partial} \widehat{N}_i^a}{\widehat{\partial} \widehat{y}^b}$  and  $\widetilde{N}_{bc}^a(\widehat{x}, \widehat{y}) = 0$ . For some  $\widehat{Y}(\widehat{u}) = \widehat{Y}^i(\widehat{u}) \widehat{\partial}_i + \widehat{Y}^a(\widehat{u}) \widehat{\partial}_a$  and  $\widehat{B}(\widehat{u}) = \widehat{B}^a(\widehat{u}) \widehat{\partial}_a$  one writes

$$\widehat{D}_Y^{(\widetilde{N})} \widehat{B} = \left[ \widehat{Y}^i \left( \frac{\widehat{\partial} \widehat{B}^a}{\widehat{\partial} \widehat{x}^i} + \widetilde{N}_{bi}^a \widehat{B}^b \right) + \widehat{Y}^b \frac{\widehat{\partial} \widehat{B}^a}{\widehat{\partial} \widehat{y}^b} \right] \frac{\widehat{\partial}}{\widehat{\partial} \widehat{y}^a}.$$



2. The d-isoconnection of Berwald type [6]

$$\widehat{\Gamma}_{\beta\gamma}^{(B)\alpha} = \left( \widehat{L}_{jk}^i, \frac{\partial \widehat{N}_k^a}{\partial \widehat{y}^b}, 0, \widehat{C}_{bc}^a \right),$$

where

$$\begin{aligned} \widehat{L}_{jk}^i(\widehat{x}, \widehat{y}) &= \frac{1}{2} \widehat{g}^{ir} \left( \frac{\partial \widehat{g}_{jk}}{\partial \widehat{x}^k} + \frac{\partial \widehat{g}_{kr}}{\partial \widehat{x}^j} - \frac{\partial \widehat{g}_{jr}}{\partial \widehat{x}^k} \right), \\ \widehat{C}_{bc}^a(\widehat{x}, \widehat{y}) &= \frac{1}{2} \widehat{h}^{ad} \left( \frac{\partial \widehat{h}_{bd}}{\partial \widehat{y}^c} + \frac{\partial \widehat{h}_{cd}}{\partial \widehat{y}^b} - \frac{\partial \widehat{h}_{bc}}{\partial \widehat{y}^d} \right), \end{aligned} \quad (4.10)$$

is **hv-isometric**, i.e.  $\widehat{D}_k^{(B)} \widehat{g}_{ij} = 0$  and  $\widehat{D}_c^{(B)} \widehat{h}_{ab} = 0$ .

3. The isocanonical d-isoconnection  $\widehat{\Gamma}^{(c)}$  is associated to a isometric  $\widehat{\mathbf{G}}$  of type (3.6)  $\widehat{\Gamma}_{\beta\gamma}^{(c)\alpha} = (\widehat{L}_{jk}^{(c)i}, \widehat{L}_{bk}^{(c)a}, \widehat{C}_{jc}^{(c)i}, \widehat{C}_{bc}^{(c)a})$ , with coefficients (see (4.10))

$$\begin{aligned} \widehat{L}_{jk}^{(c)i} &= \widehat{L}_{jk}^i, \widehat{C}_{bc}^{(c)a} = \widehat{C}_{bc}^a \\ \widehat{L}_{bi}^{(c)a} &= \widetilde{N}_{bi}^a + \frac{1}{2} \widehat{h}^{ac} \left( \frac{\partial \widehat{h}_{bc}}{\partial \widehat{x}^i} - \widetilde{N}_{bi}^d \widehat{h}_{dc} - \widetilde{N}_{ci}^d \widehat{h}_{db} \right), \\ \widehat{C}_{jc}^{(c)i} &= \frac{1}{2} \widehat{g}^{ik} \frac{\partial \widehat{g}_{jk}}{\partial \widehat{y}^c}. \end{aligned} \quad (4.11)$$

This is a isometric d-isoconnection which satisfies compatibility conditions

$$\widehat{D}_k^{(c)} \widehat{g}_{ij} = 0, \widehat{D}_c^{(c)} \widehat{g}_{ij} = 0, \widehat{D}_k^{(c)} \widehat{h}_{ab} = 0, \widehat{D}_c^{(c)} \widehat{h}_{ab} = 0.$$

4. We can consider N-adapted **isochristoffel distinguished symbols** (as in (2.5))

$$\widetilde{\Gamma}_{\beta\gamma}^\alpha = \frac{1}{2} \widehat{G}^{\alpha\tau} \left( \widehat{\delta}_\gamma \widehat{G}_{\tau\beta} + \widehat{\delta}_\beta \widehat{G}_{\tau\gamma} - \widehat{\delta}_\tau \widehat{G}_{\beta\gamma} \right), \quad (4.12)$$

which have the components of d-connection  $\widetilde{\Gamma}_{\beta\gamma}^\alpha = (\widehat{L}_{jk}^i, 0, 0, \widehat{C}_{bc}^a)$ , with  $\widehat{L}_{jk}^i$  and  $\widehat{C}_{bc}^a$  as in (4.10) if  $\widehat{G}_{\alpha\beta}$  is taken in the form (4.6).



Arbitrary isolinear isoconnections on a v-isobundle  $\widehat{\xi}$  can be also characterized by theirs deformation isotensors with respect, for instance, to d-isoconnection (4.12):

$$\widehat{\Gamma}_{\beta\gamma}^{(B)\alpha} = \widetilde{\Gamma}_{\beta\gamma}^{\alpha} + \widehat{P}_{\beta\gamma}^{(B)\alpha}, \widehat{\Gamma}_{\beta\gamma}^{(c)\alpha} = \widetilde{\Gamma}_{\beta\gamma}^{\alpha} + \widehat{P}_{\beta\gamma}^{(c)\alpha}$$

or, in general,

$$\widehat{\Gamma}_{\beta\gamma}^{\alpha} = \widetilde{\Gamma}_{\beta\gamma}^{\alpha} + \widehat{P}_{\beta\gamma}^{\alpha}, \quad (4.13)$$

where  $\widehat{P}_{\beta\gamma}^{(B)\alpha}$ ,  $\widehat{P}_{\beta\gamma}^{(c)\alpha}$  and  $\widehat{P}_{\beta\gamma}^{\alpha}$  are corresponding deformation d-isotensors of d-isoconnections.

## 5 Isotorsions and Isocurvatures

The notions of isotorsion and isocurvature were introduced in the ref. [23] on an isoriemannian spaces. In this section we reformulate these notions on isobundles provided with N-isoconnection and d-isoconnection structures.

The **isocurvature**  $\widehat{\Omega}$  of a **nonlinear isoconnection**  $\widehat{\mathbf{N}}$  in a v-isobundle  $\widehat{\xi}$  can be defined as the Nijenhuis tensor isofield  $\widehat{N}_v(\widehat{X}, \widehat{Y})$  associated to  $\widehat{\mathbf{N}}$  (this is an isotopic transform for N-curvature considered, for instance, in [14]):

$$\widehat{\Omega} = \widehat{N}_v = [\mathbf{v}\widehat{\mathbf{X}}, \mathbf{v}\widehat{\mathbf{Y}}] + \mathbf{v}[\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}] - \mathbf{v}[\mathbf{v}\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}] - \mathbf{v}[\widehat{\mathbf{X}}, \mathbf{v}\widehat{\mathbf{Y}}], \widehat{\mathbf{X}}, \widehat{\mathbf{Y}} \in \mathcal{X}(\widehat{\xi})$$

having this local representation

$$\widehat{\Omega} = \frac{1}{2} \widehat{\Omega}_{ij}^a \widehat{d}^i \wedge \widehat{d}^j \otimes \widehat{\partial}_a,$$

where

$$\widehat{\Omega}_{ij}^a = \frac{\partial \widehat{N}_i^a}{\partial \widehat{x}^j} - \frac{\partial \widehat{N}_j^a}{\partial \widehat{x}^i} + \widehat{N}_i^b \widehat{N}_{bj}^a - \widehat{N}_j^b \widehat{N}_{bi}^a. \quad (5.1)$$

The **isotorsion**  $\widehat{\mathbf{T}}$  of a **d-isoconnection**  $\widehat{\mathbf{D}}$  in  $\widehat{\xi}$  is defined by the equation

$$\widehat{\mathbf{T}}(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}) = \widehat{D}_X \widehat{\mathbf{Y}} - \widehat{D}_Y \widehat{\mathbf{X}} - [\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}]. \quad (5.2)$$

One holds the following h- and v-decompositions

$$\widehat{\mathbf{T}}(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}) = \widehat{\mathbf{T}}(\mathbf{h}\widehat{\mathbf{X}}, \mathbf{h}\widehat{\mathbf{Y}}) + \widehat{\mathbf{T}}(\mathbf{h}\widehat{\mathbf{X}}, \mathbf{v}\widehat{\mathbf{Y}}) + \widehat{\mathbf{T}}(\mathbf{v}\widehat{\mathbf{X}}, \mathbf{h}\widehat{\mathbf{Y}}) + \widehat{\mathbf{T}}(\mathbf{v}\widehat{\mathbf{X}}, \mathbf{v}\widehat{\mathbf{Y}}). \quad (5.3)$$



We consider the projections:

$$\mathbf{h}\widehat{\mathbf{T}}(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}), \mathbf{v}\widehat{\mathbf{T}}(\mathbf{h}\widehat{\mathbf{X}}, \mathbf{h}\widehat{\mathbf{Y}}), \mathbf{h}\widehat{\mathbf{T}}(\mathbf{h}\widehat{\mathbf{X}}, \mathbf{h}\widehat{\mathbf{Y}}), \dots$$

and say that, for instance,  $\mathbf{h}\widehat{\mathbf{T}}(\mathbf{h}\widehat{\mathbf{X}}, \mathbf{h}\widehat{\mathbf{Y}})$  is the  $\mathbf{h}(\mathbf{h}\mathbf{h})$ -isotorsion of  $\widehat{\mathbf{D}}$ ,  $\mathbf{v}\widehat{\mathbf{T}}(\mathbf{h}\widehat{\mathbf{X}}, \mathbf{h}\widehat{\mathbf{Y}})$  is the  $\mathbf{v}(\mathbf{h}\mathbf{h})$ -isotorsion of  $\widehat{\mathbf{D}}$  and so on.

The isotorsion (5.2) is locally determined by five d-tensor isofields, isotorsions, defined as

$$\begin{aligned} \widehat{T}_{jk}^i &= \mathbf{h}\widehat{\mathbf{T}}(\widehat{\delta}_k, \widehat{\delta}_j) \cdot \widehat{d}^i, & \widehat{T}_{jk}^a &= \mathbf{v}\widehat{\mathbf{T}}(\widehat{\delta}_k, \widehat{\delta}_j) \cdot \widehat{\delta}^a, \\ \widehat{P}_{jb}^i &= \mathbf{h}\widehat{\mathbf{T}}(\widehat{\partial}_b, \widehat{\delta}_j) \cdot \widehat{d}^i, & \widehat{P}_{jb}^a &= \mathbf{v}\widehat{\mathbf{T}}(\widehat{\partial}_b, \widehat{\delta}_j) \cdot \widehat{\delta}^a, \\ \widehat{S}_{bc}^a &= \mathbf{v}\widehat{\mathbf{T}}(\widehat{\partial}_c, \widehat{\partial}_b) \cdot \widehat{\delta}^a. \end{aligned}$$

Using formulas (4.2), (4.3), (5.1) and (5.2) we compute in explicit form the components of isotorsions (5.3) for a d-isoconnection of type (4.8) and (4.9):

$$\begin{aligned} \widehat{T}_{jk}^i &= \widehat{T}_{jk}^i = \widehat{L}_{jk}^i - \widehat{L}_{kj}^i, & \widehat{T}_{ja}^i &= \widehat{C}_{ja}^i, \widehat{T}_{aj}^i = -\widehat{C}_{ja}^i, \\ \widehat{T}_{ja}^i &= 0, \widehat{T}_{bc}^a = \widehat{S}_{bc}^a = \widehat{C}_{bc}^a - \widehat{C}_{cb}^a, \\ \widehat{T}_{ij}^a &= \frac{\widehat{\delta} N_i^a}{\widehat{\delta} \widehat{x}^j} - \frac{\widehat{\delta} \widehat{N}_j^a}{\widehat{\delta} \widehat{x}^i}, & \widehat{T}_{bi}^a &= \widehat{P}_{bi}^a = \frac{\widehat{\partial} \widehat{N}_i^a}{\widehat{\partial} \widehat{y}^b} - \widehat{L}_{bj}^a, & \widehat{T}_{ib}^a &= -\widehat{P}_{bi}^a. \end{aligned} \quad (5.4)$$

The **isocurvature**  $\widehat{\mathbf{R}}$  of a **d-isoconnection** in  $\widehat{\xi}$  is defined by the equation

$$\widehat{\mathbf{R}}(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}) \widehat{\mathbf{Z}} = \widehat{D}_X \widehat{D}_Y \widehat{\mathbf{Z}} - \widehat{D}_Y \widehat{D}_X \widehat{\mathbf{Z}} - \widehat{D}_{[X, Y]} \widehat{\mathbf{Z}}. \quad (5.5)$$

One holds the next properties for the  $\mathbf{h}$ - and  $\mathbf{v}$ -decompositions of isocurvature:

$$\begin{aligned} \mathbf{v}\widehat{\mathbf{R}}(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}) \mathbf{h}\widehat{\mathbf{Z}} &= 0, \quad \mathbf{h}\widehat{\mathbf{R}}(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}) \mathbf{v}\widehat{\mathbf{Z}} = 0, \\ \widehat{\mathbf{R}}(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}) \widehat{\mathbf{Z}} &= \mathbf{h}\widehat{\mathbf{R}}(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}) \mathbf{h}\widehat{\mathbf{Z}} + \mathbf{v}\widehat{\mathbf{R}}(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}) \mathbf{v}\widehat{\mathbf{Z}}. \end{aligned}$$

From (5.5) and the equation  $\widehat{\mathbf{R}}(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}) = -\widehat{\mathbf{R}}(\widehat{\mathbf{Y}}, \widehat{\mathbf{X}})$  we conclude that the curvature of a d-connection  $\widehat{\mathbf{D}}$  in  $\widehat{\xi}$  is completely determined by the following six d-tensor isofields:

$$\widehat{R}_{h,jk}^i = \widehat{d}^i \cdot \widehat{\mathbf{R}}(\widehat{\delta}_k, \widehat{\delta}_j) \widehat{\delta}_h, \quad \widehat{R}_{b,jk}^a = \widehat{\delta}^a \cdot \widehat{\mathbf{R}}(\widehat{\delta}_k, \widehat{\delta}_j) \widehat{\partial}_b, \quad (5.6)$$



$$\begin{aligned}\hat{P}_{j.kc}^i &= \hat{d}^i \cdot \widehat{\mathbf{R}}(\hat{\partial}_c, \hat{\partial}_k) \hat{\delta}_j, \quad \hat{P}_{b.kc}^a = \hat{\delta}^a \cdot \widehat{\mathbf{R}}(\hat{\partial}_c, \hat{\partial}_k) \hat{\partial}_b, \\ \hat{S}_{j.bc}^i &= \hat{d}^i \cdot \widehat{\mathbf{R}}(\hat{\partial}_c, \hat{\partial}_b) \hat{\delta}_j, \quad \hat{S}_{b.cd}^a = \hat{\delta}^a \cdot \widehat{\mathbf{R}}(\hat{\partial}_d, \hat{\partial}_c) \hat{\partial}_b.\end{aligned}$$

By a direct computation, using (4.2),(4.3),(4.8),(4.9) and (5.6) we get:

$$\begin{aligned}\hat{R}_{h.jk}^i &= \frac{\delta \hat{L}_{hj}^i}{\delta \hat{x}^h} - \frac{\delta \hat{L}_{hk}^i}{\delta \hat{x}^j} + \hat{L}_{hj}^m \hat{L}_{mk}^i - \hat{L}_{hk}^m \hat{L}_{mj}^i + \hat{C}_{ha}^i \hat{R}_{jk}^a, \\ \hat{R}_{b.jk}^a &= \frac{\delta \hat{L}_{bj}^a}{\delta \hat{x}^k} - \frac{\delta \hat{L}_{bk}^a}{\delta \hat{x}^j} + \hat{L}_{bj}^c \hat{L}_{ck}^a - \hat{L}_{bk}^c \hat{L}_{cj}^a + \hat{C}_{bc}^a \hat{R}_{jk}^c, \\ \hat{P}_{j.ka}^i &= \frac{\partial \hat{L}_{jk}^i}{\partial \hat{y}^k} - \left( \frac{\partial \hat{C}_{ja}^i}{\partial \hat{x}^k} + \hat{L}_{lk}^i \hat{C}_{ja}^l - \hat{L}_{jk}^l \hat{C}_{la}^i - \hat{L}_{ak}^c \hat{C}_{jc}^i \right) + \hat{C}_{jb}^i \hat{P}_{ka}^b, \\ \hat{P}_{b.ka}^c &= \frac{\partial \hat{L}_{bk}^c}{\partial \hat{y}^a} - \left( \frac{\partial \hat{C}_{ba}^c}{\partial \hat{x}^k} + \hat{L}_{dk}^c \hat{C}_{ba}^d - \hat{L}_{bk}^d \hat{C}_{da}^c - \hat{L}_{ak}^d \hat{C}_{bd}^c \right) + \hat{C}_{bd}^c \hat{P}_{ka}^d, \\ \hat{S}_{j.bc}^i &= \frac{\partial \hat{C}_{jb}^i}{\partial \hat{y}^c} - \frac{\partial \hat{C}_{jc}^i}{\partial \hat{y}^b} + \hat{C}_{jb}^h \hat{C}_{hc}^i - \hat{C}_{jc}^h \hat{C}_{hb}^i, \\ \hat{S}_{b.cd}^a &= \frac{\partial \hat{C}_{bc}^a}{\partial \hat{y}^d} - \frac{\partial \hat{C}_{bd}^a}{\partial \hat{y}^c} + \hat{C}_{bc}^e \hat{C}_{ed}^a - \hat{C}_{bd}^e \hat{C}_{ec}^a.\end{aligned}\tag{5.7}$$

We note that isotorsions (5.4) and isocurvatures (5.7) can be computed by particular cases of d-isoconnections when d-isoconnections (4.11), or (4.12) are used instead of (4.8) and (4.9). The above presented formulas are similar to (2.8),(2.9) and (2.10) being distinguished (in the case of locally anisotropic and inhomogeneous isospaces) by N-isoconnection structure.

For our further considerations it is useful to compute deformations of isotorsion (5.2) and isocurvature (5.5) under deformations of d-connections (4.13). Putting the splitting (4.13),  $\hat{\Gamma}^\alpha_{\beta\gamma} = \tilde{\Gamma}^\alpha_{\beta\gamma} + \hat{P}^\alpha_{\beta\gamma}$ , into (5.2) and (5.5) we can express isotorsion  $\hat{T}^\alpha_{\beta\gamma}$  and isocurvature  $\hat{R}^\alpha_{\beta\gamma\delta}$  of a d-isoconnection  $\hat{\Gamma}^\alpha_{\beta\gamma}$  as respective deformations of isotorsion  $\tilde{T}^\alpha_{\beta\gamma}$  and isocurvature  $\tilde{R}^\alpha_{\beta\gamma\delta}$  for connection  $\tilde{\Gamma}^\alpha_{\beta\gamma}$  :

$$T^\alpha_{\beta\gamma} = \tilde{T}^\alpha_{\beta\gamma} + \hat{T}^\alpha_{\beta\gamma}$$

and

$$R^\alpha_{\beta\gamma\delta} = \tilde{R}^\alpha_{\beta\gamma\delta} + \hat{R}^\alpha_{\beta\gamma\delta},$$



where

$$\tilde{T}_{\beta\gamma}^\alpha = \tilde{\Gamma}_{\beta\gamma}^\alpha - \tilde{\Gamma}_{\gamma\beta}^\alpha + w^\alpha_{\gamma\delta}, \quad \ddot{T}_{\beta\gamma}^\alpha = \ddot{\Gamma}_{\beta\gamma}^\alpha - \ddot{\Gamma}_{\gamma\beta}^\alpha,$$

and

$$\begin{aligned} \tilde{R}_{\beta\cdot\gamma\delta}^\alpha &= \delta_\delta \tilde{\Gamma}_{\beta\gamma}^\alpha - \delta_\gamma \tilde{\Gamma}_{\beta\delta}^\alpha + \tilde{\Gamma}_{\beta\gamma}^\varphi \tilde{\Gamma}_{\varphi\delta}^\alpha - \tilde{\Gamma}_{\beta\delta}^\varphi \tilde{\Gamma}_{\varphi\gamma}^\alpha + \tilde{\Gamma}_{\beta\varphi}^\alpha w^\varphi_{\gamma\delta}, \\ \ddot{R}_{\beta\cdot\gamma\delta}^\alpha &= \ddot{D}_\delta P^\alpha_{\beta\gamma} - \ddot{D}_\gamma P^\alpha_{\beta\delta} + P^\varphi_{\beta\gamma} P^\alpha_{\varphi\delta} - P^\varphi_{\beta\delta} P^\alpha_{\varphi\gamma} + P^\alpha_{\beta\varphi} w^\varphi_{\gamma\delta}. \end{aligned}$$

## 5.1 Isobianchi and Isoricci Identities

The isobianchi and isoricci identities were first studied by Santilli [23] on an isoriemannian space. On spaces with N-connection structures the general formulas for Bianchi and Ricci identities (for osculator and vector bundles, generalized Lagrange and Finsler geometry) have been considered by Miron and Anastasiei [14] and Miron and Atanasiu [15]. We have extended the Miron–Anastasiei–Atanasiu constructions for superspaces with local and higher order anisotropy in refs. [31, 30, 33]. The purpose of this section is to consider distinguished isobianchi and isorichi for vector isobundles.

The isotorsion and isocurvature of every linear isoconnection  $\widehat{D}$  on a v-isobundle satisfy the following **generalized isobianchi identities**:

$$\sum [(\widehat{D}_{\widehat{X}} \widehat{T})(\widehat{Y}, \widehat{Z}) - \widehat{R}(\widehat{X}, \widehat{Y}) \widehat{Z} + \widehat{T}(\widehat{T}(\widehat{X}, \widehat{Y}), \widehat{Z})] = 0, \quad (5.8)$$

$$\sum [(\widehat{D}_{\widehat{X}} \widehat{R})(\widehat{U}, \widehat{Y}, \widehat{Z}) + \widehat{R}(\widehat{T}(\widehat{X}, \widehat{Y}) \widehat{Z}) \widehat{U}] = 0,$$

where  $\sum$  means the respective cyclic sum over  $\widehat{X}, \widehat{Y}, \widehat{Z}$  and  $\widehat{U}$ . Using the property that

$$v(\widehat{D}_X \widehat{R})(\widehat{U}, \widehat{Y}, h\widehat{Z}) = 0, \quad h(\widehat{D}_X \widehat{R})(\widehat{U}, \widehat{Y}, v\widehat{Z}) = 0,$$

the identities (5.8) become

$$\begin{aligned} &\sum [h(\widehat{D}_X \widehat{T})(\widehat{Y}, \widehat{Z}) - h\widehat{R}(\widehat{X}, \widehat{Y}) \widehat{Z} + \\ &h\widehat{T}(h\widehat{T}(\widehat{X}, \widehat{Y}), \widehat{Z}) + h\widehat{T}(v\widehat{T}(\widehat{X}, \widehat{Y}), \widehat{Z})] = 0, \\ &\sum [v(\widehat{D}_X \widehat{T})(\widehat{Y}, \widehat{Z}) - v\widehat{R}(\widehat{X}, \widehat{Y}) \widehat{Z} + \\ &v\widehat{T}(h\widehat{T}(\widehat{X}, \widehat{Y}), \widehat{Z}) + v\widehat{T}(v\widehat{T}(\widehat{X}, \widehat{Y}), \widehat{Z})] = 0, \end{aligned} \quad (5.9)$$



$$\begin{aligned} \sum [h(\widehat{D}_X \widehat{R})(\widehat{U}, \widehat{Y}, \widehat{Z}) + h\widehat{R}(h\widehat{T}(\widehat{X}, \widehat{Y}), \widehat{Z})\widehat{U} + h\widehat{R}(v\widehat{T}(\widehat{X}, \widehat{Y}), \widehat{Z})\widehat{U}] &= 0, \\ \sum [v(\widehat{D}_X \widehat{R})(\widehat{U}, \widehat{Y}, \widehat{Z}) + v\widehat{R}(h\widehat{T}(\widehat{X}, \widehat{Y}), \widehat{Z})\widehat{U} + v\widehat{R}(v\widehat{T}(\widehat{X}, \widehat{Y}), \widehat{Z})\widehat{U}] &= 0. \end{aligned}$$

The local adapted form of these identities is obtained by inserting in (5.9) the necessary values of triples  $(\widehat{X}, \widehat{Y}, \widehat{Z})$ ,  $(= (\widehat{\delta}_i, \widehat{\delta}_k, \widehat{\delta}_l), \text{ or } (\widehat{\partial}_d, \widehat{\partial}_c, \widehat{\partial}_b),)$  and putting successively  $\widehat{U} = \widehat{\delta}_h$  and  $\widehat{U} = \widehat{\partial}_a$ . Taking into account (4.2), (4.3) and (5.9) we obtain:

$$\begin{aligned} \sum [\widehat{T}_{jk|h}^i + \widehat{T}_{jk}^m \widehat{T}_{hm}^j + \widehat{R}_{jk}^a \widehat{C}_{ha}^i - \widehat{R}_{j\cdot kh}^i] &= 0, \quad (5.10) \\ \sum [\widehat{R}_{jk|h}^a + \widehat{T}_{jk}^m \widehat{R}_{hm}^a + \widehat{R}_{jk}^b \widehat{P}_{hb}^a] &= 0, \\ \widehat{C}_{jb|k}^i - \widehat{C}_{kb|j}^i - \widehat{T}_{jk|b}^i + \widehat{C}_{jb}^m \widehat{T}_{km}^i - \widehat{C}_{kb}^m \widehat{T}_{jm}^i + \widehat{T}_{jk}^m \widehat{C}_{mb}^i + \\ \widehat{P}_{jb}^d \widehat{C}_{kd}^i - \widehat{P}_{kb}^d \widehat{C}_{jd}^i + \widehat{P}_{j\cdot kb}^i - \widehat{P}_{k\cdot jb}^i &= 0, \\ \widehat{P}_{jb|k}^a - \widehat{P}_{kb|j}^a - \widehat{R}_{jk\perp b}^a + \widehat{C}_{jb}^m \widehat{R}_{km}^a - \widehat{C}_{kb}^m \widehat{R}_{jm}^a + \\ \widehat{T}_{jk}^m \widehat{P}_{mb}^a + \widehat{P}_{db}^d \widehat{P}_{kd}^a - \widehat{P}_{kb}^d \widehat{P}_{jd}^a - \widehat{R}_{jk}^d \widehat{S}_{bd}^a + \widehat{R}_{b\cdot jk}^a &= 0, \\ \widehat{C}_{jb\perp c}^i - \widehat{C}_{jc\perp b}^i + \widehat{C}_{jc}^m \widehat{C}_{mb}^i - \\ \widehat{C}_{jb}^m \widehat{C}_{mc}^i + \widehat{S}_{bc}^d \widehat{C}_{jd}^i - \widehat{S}_{j\cdot bc}^i &= 0, \\ \widehat{P}_{jb\perp c}^a - \widehat{P}_{jc\perp b}^a + \widehat{S}_{bc|j}^a + \widehat{C}_{jc}^m \widehat{P}_{mb}^a - \widehat{C}_{jb}^m \widehat{P}_{mc}^a + \\ \widehat{P}_{jb}^d \widehat{S}_{cd}^a - \widehat{P}_{jc}^d \widehat{S}_{bd}^a + \widehat{S}_{bc}^d \widehat{P}_{jd}^a + \widehat{P}_{b\cdot jc}^a - \widehat{P}_{c\cdot jb}^a &= 0, \\ \sum [\widehat{S}_{bc\perp d}^a + \widehat{S}_{bc}^f \widehat{S}_{df}^a - \widehat{S}_{f\cdot cd}^a] &= 0, \\ \sum [\widehat{R}_{k\cdot hj|l}^i - \widehat{T}_{hj}^m \widehat{R}_{lm}^i - \widehat{R}_{hj}^a \widehat{P}_{k\cdot la}^i] &= 0, \\ \sum [\widehat{R}_{d\cdot hj|l}^a - \widehat{T}_{hj}^m \widehat{R}_{d\cdot lm}^a - \widehat{R}_{hj}^c \widehat{P}_{d\cdot lc}^a] &= 0, \\ \widehat{P}_{k\cdot jd|l}^i - \widehat{P}_{k\cdot ld|j}^i + R_{kj}^i \widehat{P}_{ld}^i + C_{ld}^m R_{jm}^i - C_{jd}^m R_{lm}^i - \\ \widehat{T}_{jl}^m \widehat{P}_{k\cdot md}^i + \widehat{P}_{ld}^a \widehat{P}_{k\cdot jl}^i - \widehat{P}_{jd}^a \widehat{P}_{k\cdot la}^i - \widehat{R}_{jl}^a \widehat{S}_{k\cdot ad}^i &= 0, \\ \widehat{P}_{c\cdot jd|l}^a - \widehat{P}_{c\cdot ld|j}^a + \widehat{R}_{c\cdot lj|d}^a + \widehat{C}_{ld}^m \widehat{R}_{c\cdot jm}^a - \widehat{C}_{jd}^m \widehat{R}_{c\cdot lm}^a - \\ \widehat{T}_{jl}^m \widehat{P}_{c\cdot md}^a + \widehat{P}_{ld}^f \widehat{P}_{c\cdot jf}^a - \widehat{P}_{jd}^f \widehat{P}_{c\cdot lf}^a - \widehat{R}_{jl}^f \widehat{S}_{c\cdot fd}^a &= 0, \\ \widehat{P}_{k\cdot jd\perp c}^i - \widehat{P}_{k\cdot jc\perp d}^i + \widehat{S}_{k\cdot dc|j}^i + \widehat{C}_{jd}^m \widehat{P}_{k\cdot mc}^i - \widehat{C}_{jc}^m \widehat{P}_{k\cdot md}^i + \end{aligned}$$



$$\begin{aligned}
& \hat{P}^a_{jc} \hat{S}^i_{k \cdot da} - \hat{P}^a_{jd} \hat{S}^i_{k \cdot ca} + \hat{S}^a_{cd} \hat{P}^i_{k \cdot ja} = 0, \\
& \hat{P}^a_{b \cdot jd \perp c} - \hat{P}^a_{b \cdot jc \perp d} + \hat{S}^a_{cd|j} + \hat{C}^m_{jd} \hat{P}^a_{b \cdot mc} - \\
& \hat{C}^m_{jc} \hat{P}^a_{b \cdot md} + \hat{P}^f_{jc} \hat{S}^a_{b \cdot df} - \hat{P}^f_{jd} \hat{S}^a_{b \cdot cf} + \hat{S}^f_{cd} \hat{P}^a_{b \cdot jf} = 0, \\
& \sum_{[b,c,d]} \hat{S}^i_{k \cdot bc \perp d} - \hat{S}^a_{bc} \hat{S}^i_{k \cdot da} \\
& \sum_{[b,c,d]} [\hat{S}^a_{f \cdot bc \perp d} - \hat{S}^e_{bc} \hat{S}^a_{f \cdot de}] = 0,
\end{aligned}$$

where, for instance,  $\sum_{[b,c,d]}$  means the cyclic sum over indices  $b, c, d$ .

Identities (5.10) are isotopic generalizations of the corresponding formulas presented in [14], or equivalently, an extension of Santilli's [23] formulas to the case of d-isoconnections.

As a consequence of a corresponding rearrangement of (5.9) we obtain the **isoricci identities** (for simplicity we establish them only for distinguished vector isofields, although they may be written for every distinguished tensor isofield):

$$\begin{aligned}
\widehat{D}^{(h)}_{[X} \widehat{D}^{(h)}_{Y]} h \widehat{Z} &= \widehat{R}(h \widehat{X}, h \widehat{Y}) h \widehat{Z} + \widehat{D}^{(h)}_{[hX, hY]} h \widehat{Z} + \widehat{D}^{(v)}_{[hX, hY]} h \widehat{Z}, \quad (5.11) \\
\widehat{D}^{(v)}_{[X} \widehat{D}^{(h)}_{Y]} h \widehat{Z} &= \widehat{R}(v \widehat{X}, h \widehat{Y}) h \widehat{Z} + \widehat{D}^{(h)}_{[vX, hY]} h \widehat{Z} + \widehat{D}^{(v)}_{[vX, hY]} h \widehat{Z}, \\
\widehat{D}^{(v)}_{[X} \widehat{D}^{(v)}_{Y]} h \widehat{Z} &= \widehat{R}(v \widehat{X}, v \widehat{Y}) h \widehat{Z} + \widehat{D}^{(v)}_{[vX, vY]} h \widehat{Z}
\end{aligned}$$

and

$$\begin{aligned}
\widehat{D}^{(h)}_{[X} \widehat{D}^{(h)}_{Y]} v \widehat{Z} &= \widehat{R}(h \widehat{X}, h \widehat{Y}) v \widehat{Z} + \widehat{D}^{(h)}_{[hX, hY]} v \widehat{Z} + \widehat{D}^{(v)}_{[hX, hY]} v \widehat{Z}, \quad (5.12) \\
\widehat{D}^{(v)}_{[X} \widehat{D}^{(h)}_{Y]} v \widehat{Z} &= \widehat{R}(v \widehat{X}, h \widehat{Y}) v \widehat{Z} + \widehat{D}^{(v)}_{[vX, hY]} v \widehat{Z} + \widehat{D}^{(v)}_{[vX, hY]} v \widehat{Z}, \\
\widehat{D}^{(v)}_{[X} \widehat{D}^{(v)}_{Y]} v \widehat{Z} &= \widehat{R}(v \widehat{X}, v \widehat{Y}) v \widehat{Z} + \widehat{D}^{(v)}_{[vX, vY]} v \widehat{Z}.
\end{aligned}$$

For  $\widehat{X} = \widehat{X}^i(\widehat{u}) \frac{\widehat{\delta}}{\delta x^i} + \widehat{X}^a(\widehat{u}) \frac{\widehat{\partial}}{\partial x^a}$  and (4.2), (4.3), (5.4) and (5.7) we can express respectively identities (5.11) and (5.12) in this form:

$$\begin{aligned}
\widehat{X}^a_{|k|l} - \widehat{X}^a_{|l|k} &= \widehat{R}^a_{B \cdot kl} \widehat{X}^b - \widehat{T}^h_{kl} \widehat{X}^a_{|h} - \widehat{R}^b_{kl} \widehat{X}^a_{\perp b}, \\
\widehat{X}^i_{|k \perp d} - \widehat{X}^i_{\perp d|k} &= \widehat{P}^i_{h \cdot kd} \widehat{X}^h - \widehat{C}^h_{kd} \widehat{X}^i_{|h} - \widehat{P}^a_{kd} \widehat{X}^i_{\perp a},
\end{aligned}$$



$$\widehat{X}^i_{\perp b \perp c} - \widehat{X}^i_{\perp c \perp b} = \widehat{S}^i_{h \cdot bc} \widehat{X}^h - \widehat{S}^a_{bc} \widehat{X}^i_{\perp a}$$

and

$$\begin{aligned} \widehat{X}^a_{|k|l} - \widehat{X}^a_{|l|k} &= \widehat{R}^a_{b \cdot kl} \widehat{X}^b - \widehat{T}^h_{kl} \widehat{X}^a_{|h} - \widehat{R}^b_{kl} \widehat{X}^a_{\perp b}, \\ \widehat{X}^a_{|k \perp b} - \widehat{X}^a_{\perp b|k} &= \widehat{P}^a_{b \cdot kc} \widehat{X}^c - \widehat{C}^h_{kb} \widehat{X}^a_{|h} - \widehat{P}^d_{kb} \widehat{X}^a_{\perp d}, \\ \widehat{X}^a_{\perp b \perp c} - \widehat{X}^a_{\perp c \perp b} &= \widehat{S}^a_{d \cdot bc} \widehat{X}^d - \widehat{S}^d_{bc} \widehat{X}^a_{\perp d}. \end{aligned}$$

For some considerations it is useful to use an alternative way of definition isotorsion (5.2) and isocurvature (5.5) by using the commutator

$$\widehat{\Delta}_{\alpha\beta} \doteq \widehat{\nabla}_{\alpha} \widehat{\nabla}_{\beta} - \widehat{\nabla}_{\beta} \widehat{\nabla}_{\alpha} = 2\widehat{\nabla}_{[\alpha} \widehat{\nabla}_{\beta]}. \quad (5.13)$$

For components (5.13) of d-isotorsion we have

$$\widehat{\Delta}_{\alpha\beta} \widehat{f} = \widehat{T}^{\gamma}_{\cdot\alpha\beta} \widehat{\nabla}_{\gamma} \widehat{f}$$

for every scalar function  $\widehat{f}$  on  $\widehat{\xi}$ . Curvature can be introduced as an operator acting on arbitrary d-isovector  $\widehat{V}^{\delta}$ :

$$(\widehat{\Delta}_{\alpha\beta} - \widehat{T}^{\gamma}_{\cdot\alpha\beta} \widehat{\nabla}_{\gamma}) \widehat{V}^{\delta} = \widehat{R}^{\delta}_{\gamma \cdot \alpha\beta} \widehat{V}^{\gamma}$$

(in this work we are following conventions similar to Miron and Anastasiei [14] on d-isotensors; we can obtain corresponding Penrose and Rindler abstract index formulas [17] just for a trivial N-connection structure and by changing denotations for components of isotorsion and isocurvature in this manner:  $T^{\gamma}_{\cdot\alpha\beta} \rightarrow T^{\gamma}_{\alpha\beta}$  and  $R^{\delta}_{\gamma \cdot \alpha\beta} \rightarrow R^{\delta}_{\alpha\beta\gamma}$ ).

## 5.2 Structure Equations of a d-Isoconnection

Let us, for instance, consider d-tensor isofield:

$$\widehat{t} = \widehat{t}^i_a \widehat{\delta}_I \otimes \widehat{\delta}^a.$$

We introduce the so-called d-connection 1-forms  $\omega^i_j$  and  $\tilde{\omega}^a_b$  as

$$\widehat{D}\widehat{t} = (\widehat{D}\widehat{t}^i_a) \widehat{\delta}_I \otimes \widehat{\delta}^a$$

with

$$\widehat{D}\widehat{t}^i_a = d\widehat{t}^i_a + \omega^i_j \widehat{t}^j_a - \tilde{\omega}^b_a \widehat{t}^i_b = \widehat{t}^i_{a|j} d\widehat{x}^j + \widehat{t}^I_{a \perp b} \widehat{\delta}^b.$$



For the d-isoconnection 1-forms of a d-isoconnection  $\widehat{D}$  on  $\widehat{\xi}$  defined by  $\omega_j^i$  and  $\tilde{\omega}_b^a$  one holds the following **structure isoequations**:

$$d(\widehat{d}^i) - \widehat{d}^h \wedge \omega_h^i = -\widehat{\Omega}^i,$$

$$d(\widehat{\delta}^a) - \widehat{\delta}^a \wedge \omega_b^a = -\widehat{\Omega}^a,$$

$$d\omega_j^i - \omega_j^h \wedge \omega_h^i = -\widehat{\Omega}_j^i,$$

$$d\omega_b^a - \omega_b^c \wedge \omega_c^a = -\widehat{\Omega}_b^a,$$

in which the isotorsion 2-forms  $\widehat{\Omega}^i$  and  $\widehat{\Omega}^i$  are given respectively by formulas:

$$\widehat{\Omega}^i = \frac{1}{2} \widehat{T}^i_{jk} \widehat{d}^j \wedge \widehat{d}^k + \frac{1}{2} \widehat{C}^i_{jk} \widehat{d}^j \wedge \widehat{\delta}^c,$$

$$\widehat{\Omega}^a = \frac{1}{2} \widehat{R}^a_{jk} \widehat{d}^j \wedge \widehat{d}^k + \frac{1}{2} \widehat{P}^a_{jc} \widehat{d}^j \wedge \widehat{\delta}^c + \frac{1}{2} \widehat{S}^s_{bc} \widehat{\delta}^b \wedge \widehat{\delta}^c,$$

and

$$\widehat{\Omega}_j^i = \frac{1}{2} \widehat{R}_{j\cdot kh}^i \widehat{d}^k \wedge \widehat{d}^h + \frac{1}{2} \widehat{P}_{j\cdot kc}^i \widehat{d}^k \wedge \widehat{\delta}^c + \frac{1}{2} \widehat{S}_{j\cdot kc}^i \widehat{\delta}^b \wedge \widehat{\delta}^c,$$

$$\widehat{\Omega}_b^a = \frac{1}{2} \widehat{R}_{b\cdot kh}^a \widehat{d}^k \wedge \widehat{d}^h + \frac{1}{2} \widehat{P}_{b\cdot kc}^a \widehat{d}^k \wedge \widehat{\delta}^c + \frac{1}{2} \widehat{S}_{b\cdot cd}^a \widehat{\delta}^c \wedge \widehat{\delta}^d$$

The just presented formulas are very similar to those for usual locally anisotropic spaces [14] but in our case they are written for isotopic values and generalize the isoriemannian Santilli's formulas [23].

## 6 The Isogeometry of Tangent Isobundles

The aim of this section is to formulate some results in the isogeometry of tangent isobundle, t-isobundle,  $\widehat{TM}$  and to use them in order to develop the geometry of Finsler and Lagrange isospaces.

All results presented in the preceding section on v-isobundles provided with N-isoconnection, d-isoconnection and isometric structures hold good for  $\widehat{TM}$ . In this case the dimension of the base isospace and of typical isofibre coincides and we can write locally, for instance, isovectors as

$$\widehat{X} = \widehat{X}^i \widehat{\delta}_i + \widehat{Y}^i \widehat{\partial}_i = \widehat{X}^i \widehat{\delta}_i + \widehat{Y}^{(i)} \widehat{\partial}_{(i)},$$



where  $\hat{u}^\alpha = (\hat{x}^i, \hat{y}^j) = (\hat{x}^i, \hat{y}^{(j)})$ .

On t-isobundles we can define a global map

$$\hat{J} : \mathcal{X}(\widehat{TM}) \rightarrow \mathcal{X}(\widehat{TM}) \quad (6.1)$$

which does not depend on N-isoconnection structure:

$$\hat{J}\left(\frac{\hat{\delta}}{\delta \hat{x}^i}\right) = \frac{\hat{\partial}}{\partial \hat{y}^i}$$

and

$$\hat{J}\left(\frac{\hat{\partial}}{\partial \hat{y}^i}\right) = 0.$$

This endomorphism is called the **natural (or canonical) almost tangent isostructure** on  $\widehat{TM}$ ; it has the properties:

$$1) \hat{J}^2 = 0, \quad 2) Im \hat{J} = Ker \hat{J} = V\widehat{TM}$$

and 3) the **Nigenhuis isotensor**,

$$N_J(\widehat{X}, \widehat{Y}) = [J\widehat{X}, J\widehat{Y}] - J[J\widehat{X}, \widehat{Y}] - J[\widehat{X}, J\widehat{Y}]$$

$$(\widehat{X}, \widehat{Y} \in \mathcal{X}(\widehat{TM}))$$

identically vanishes, i.e. the natural almost tangent isostructure  $J$  on  $\widehat{TM}$  is isointegrable.

## 6.1 Notions of Generalized Isolagrange, Isolagrange and Isofinsler Spaces

Let  $\widehat{M}$  be a isosmooth  $(2n)$ -dimensional isomanifold and  $(\widehat{TM}, \tau, \widehat{M})$  its t-isobundle. For isospaces we define a **generalized isolagrange space**, GIL-space, as a pair  $G\widehat{L}^{n,m} = (\widehat{M}, \hat{g}_{ij}(\hat{x}, \hat{y}))$ , where  $\hat{g}_{ij}(\hat{x}, \hat{y})$  is a d-tensor isofield on  $\tilde{TM} = \widehat{TM} - \{0\}$ , of isorank  $(2n)$ , and is called as the fundamental d-isotensor, or metric d-isotensor, of GIL-space.

Let denote as a normal d-isoconnection that defined by using  $N$  and being adapted to the almost tangent isostructure (6.1) as  $\widehat{D}\Gamma = (\hat{L}^a_{jk}, \hat{C}^a_{jk})$ .



This d-isoconnection is compatible with isometric  $\widehat{g}_{ij}(\widehat{x}, \widehat{y})$  if  $\widehat{g}_{ij|k} = 0$  and  $\widehat{g}_{ij\perp k} = 0$ .

There exists an unique d-isoconnection  $C\widehat{\Gamma}(N)$  which is compatible with  $\widehat{g}_{ij}(\widehat{u})$  and has vanishing isotorsions  $\widehat{T}^i_{jk}$  and  $\widehat{S}^i_{jk}$  (see formulas (5.4) rewritten for t-isobundles). This isoconnection, depending only on  $\widehat{g}_{ij}(\widehat{u})$  and  $\widehat{N}^i_j(\widehat{u})$  is called the canonical metric d-isoconnection of GIL-space. It has coefficients

$$\begin{aligned}\widehat{L}^i_{jk} &= \frac{1}{2}\widehat{g}^{ih}(\widehat{\delta}_j\widehat{g}_{hk} + \widehat{\delta}_h\widehat{g}_{jk} - \widehat{\delta}_h\widehat{g}_{jk}), \\ \widehat{C}^i_{jk} &= \frac{1}{2}\widehat{g}^{ih}(\widehat{\partial}_j\widehat{g}_{hk} + \widehat{\partial}_h\widehat{g}_{jk} - \widehat{\partial}_h\widehat{g}_{jk}).\end{aligned}\tag{6.2}$$

Of course, metric d-isoconnections different from  $C\widehat{\Gamma}(N)$  may be found. For instance, there is a unique normal d-isoconnection  $\widehat{D}\Gamma(N) = (\bar{L}^i_{jk}, \bar{C}^i_{jk})$  which is metric and has a priori given isotorsions  $\widehat{T}^i_{jk}$  and  $\widehat{S}^i_{jk}$ . The coefficients of  $\widehat{D}\Gamma(N)$  are the following ones:

$$\begin{aligned}\bar{L}^i_{jk} &= \widehat{L}^i_{jk} - \frac{1}{2}\widehat{g}^{ih}(\widehat{g}_{jr}\widehat{T}^r_{hk} + \widehat{g}_{kr}\widehat{T}^r_{hj} - \widehat{g}_{hr}\widehat{T}^r_{kj}), \\ \bar{C}^i_{jk} &= \widehat{C}^i_{jk} - \frac{1}{2}\widehat{g}^{ih}(\widehat{g}_{jr}\widehat{S}^r_{hk} + \widehat{g}_{kr}\widehat{S}^r_{hj} - \widehat{g}_{hr}\widehat{S}^r_{kj}),\end{aligned}$$

where  $\widehat{L}^i_{jk}$  and  $\widehat{C}^i_{jk}$  are the same as for the  $C\widehat{\Gamma}(N)$ -isoconnection (6.2).

The Lagrange spaces were introduced in order to geometrize the concept of Lagrangian in mechanics (the Lagrange geometry is studied in details, see also basic references, in Miron and Anastasiei [14]). For isospaces we present this generalization:

A **isolagrange space**, IL-space,  $\widehat{L}^n = (\widehat{M}, \widehat{g}_{ij})$ , is defined as a particular case of GIL-space when the d-isometric on  $\widehat{M}$  can be expressed as

$$\widehat{g}_{ij}(\widehat{u}) = \frac{1}{2} \frac{\widehat{\partial}^2 \mathcal{L}}{\partial \widehat{y}^i \partial \widehat{y}^j},\tag{6.3}$$

where  $\mathcal{L} : \widehat{T}\widehat{M} \rightarrow \widehat{\Lambda}$ , is a isodifferentiable function called a iso-Lagrangian on  $\widehat{M}$ .

Now we consider the isotopic extension of the Finsler space:

A **isofinsler isometric** on  $\widehat{M}$  is a function  $F_S : \widehat{T}\widehat{M} \rightarrow \widehat{\Lambda}$  having the properties:



1. The restriction of  $F_S$  to  $T\tilde{M} = \widehat{T\tilde{M}} \setminus \{0\}$  is of the class  $G^\infty$  and  $F$  is only isosmooth on the image of the null cross-section in the  $t$ -isobundle to  $\widehat{M}$ .
2. The restriction of  $\widehat{F}$  to  $T\tilde{M}$  is positively homogeneous of degree 1 with respect to  $(\widehat{y}^i)$ , i.e.  $\widehat{F}(\widehat{x}, \widehat{\lambda}\widehat{y}) = \widehat{\lambda}\widehat{F}(\widehat{x}, \widehat{y})$ , where  $\widehat{\lambda}$  is a real positive number.
3. The restriction of  $\widehat{F}$  to the even subspace of  $T\tilde{M}$  is a positive function.
4. The quadratic form on  $\Lambda^n$  with the coefficients

$$\widehat{g}_{ij}(\widehat{u}) = \frac{1}{2} \frac{\widehat{\partial}^2 \widehat{F}^2}{\widehat{\partial} \widehat{y}^i \widehat{\partial} \widehat{y}^j} \quad (6.4)$$

defined on  $T\tilde{M}$  is nondegenerate.

A pair  $\widehat{F}^n = (\widehat{M}, \widehat{F})$  which consists from a continuous isomanifold  $\widehat{M}$  and a isofinsler isometric is called a **isofinsler space**, IF-space.

It's obvious that IF-spaces form a particular class of IL-spaces with iso-Lagrangian  $\mathcal{L} = \widehat{F}^2$  and a particular class of GIL-spaces with metrics of type (6.4).

For a IF-space we can introduce the isotopic variant of nonlinear Cartan connection [14]:

$$\widehat{N}_j^i(\widehat{x}, \widehat{y}) = \frac{\widehat{\partial}}{\widehat{\partial} \widehat{y}^j} \widehat{G}^{*I},$$

where

$$\widehat{G}^{*i} = \frac{1}{4} \widehat{g}^{*ij} \left( \frac{\widehat{\partial}^2 \varepsilon}{\widehat{\partial} \widehat{y}^i \widehat{\partial} \widehat{x}^k} \widehat{y}^k - \frac{\widehat{\partial} \varepsilon}{\widehat{\partial} \widehat{x}^j} \right), \quad \varepsilon(\widehat{u}) = \widehat{g}_{ij}(\widehat{u}) \widehat{y}^i \widehat{y}^j,$$

and  $\widehat{g}^{*ij}$  is inverse to  $\widehat{g}_{ij}^*(\widehat{u}) = \frac{1}{2} \frac{\widehat{\partial}^2 \varepsilon}{\widehat{\partial} \widehat{y}^i \widehat{\partial} \widehat{y}^j}$ . In this case the coefficients of canonical metric d-isoconnection (6.2) gives the isotopic variants of coefficients of the Cartan connection of Finsler spaces. A similar remark applies to the isolagrange spaces.

## 6.2 The Isotopic Almost Hermitian Model of GIL-Spaces

Consider a GIL-space endowed with the canonical metric d-isoconnection  $C\widehat{\Gamma}(N)$ . Let  $\widehat{\delta}_\alpha = (\widehat{\delta}_\alpha, \widehat{\partial}_I)$  be a usual adapted frame (4.2) on  $TM$  and  $\widehat{\delta}^\alpha =$



$(\widehat{\partial}^I, \widehat{\delta}^I)$  its dual, see (4.3). The linear operator

$$\widehat{F} : \Xi(T\widetilde{M}) \rightarrow \Xi(T\widetilde{M}),$$

acting on  $\widehat{\delta}_\alpha$  by  $\widehat{F}(\widehat{\delta}_i) = -\widehat{\partial}_i$ ,  $\widehat{F}(\widehat{\partial}_i) = \widehat{\delta}_i$ , defines an almost complex isostructure on  $T\widetilde{M}$ . We shall obtain a complex isostructure if and only if the even component of the horizontal distribution  $\widehat{N}$  is integrable. For isospaces, in general with even and odd components, we write the isotopic almost Hermitian property (almost Hermitian isostructure) as

$$\widehat{F}_\beta^\alpha \widehat{F}_\delta^\beta = -\delta_\beta^\alpha.$$

The isometric  $\widehat{g}_{ij}(\widehat{x}, \widehat{y})$  on GIL-spaces induces on  $T\widehat{M}$  the following isometric:

$$\widehat{G} = \widehat{g}_{ij}(\widehat{u}) \widehat{d}\widehat{x}^i \otimes \widehat{d}\widehat{x}^j + \widehat{g}_{ij}(\widehat{u}) \widehat{\delta}\widehat{y}^i \otimes \widehat{\delta}\widehat{y}^j. \quad (6.5)$$

We can verify that pair  $(\widehat{G}, \widehat{F})$  is an almost Hermitian isostructure on  $T\widehat{M}$  with the associated supersymmetric 2-form

$$\widehat{\theta} = \widehat{g}_{ij}(\widehat{x}, \widehat{y}) \widehat{\delta}\widehat{y}^i \wedge \widehat{d}\widehat{x}^j.$$

The almost Hermitian isospace  $\widehat{H}^{2n} = (T\widehat{M}, \widehat{G}, \widehat{F})$ , provided with a isometric of type (6.5) is called the lift on  $T\widehat{M}$ , or the almost Hermitian isomodel, of GIL-space  $G\widehat{L}^n$ . We say that a linear isoconnection  $\widehat{D}$  on  $T\widehat{M}$  is almost Hermitian isotopic of Lagrange type if it preserves by parallelism the vertical distribution  $V$  and is compatible with the almost Hermitian isostructure  $(\widehat{G}, \widehat{F})$ , i.e.

$$\widehat{D}_X \widehat{G} = 0, \quad \widehat{D}_X \widehat{F} = 0, \quad (6.6)$$

for every  $X \in \widehat{\mathcal{X}}(T\widehat{M})$ .

There exists an unique almost Hermitian isoconnection of Lagrange type  $\widehat{D}^{(c)}$  having h(hh)- and v(vv)-isotorsions equal to zero. We can prove (similarly as in [14]) that coefficients  $(\widehat{L}_{jk}^i, \widehat{C}_{jk}^i)$  of  $\widehat{D}^{(c)}$  in the adapted basis  $(\widehat{\delta}_i, \widehat{\delta}_j)$  are just the coefficients (6.2) of the canonical metric d-isoconnection  $C\widehat{\Gamma}(N)$  of the GIL-space  $G\widehat{L}^n$ . Inversely, we can say that  $C\widehat{\Gamma}(N)$ -connection determines on  $T\widetilde{M}$  and isotopic almost Hermitian connection of Lagrange



type with vanishing  $h(hh)$ - and  $v(vv)$ -isotorsions. If instead of GIL-space isometric  $\hat{g}_{ij}$  in (6.4) the isolagrange (or isofinsler) isometric (6.2) (or (6.3)) is taken, we obtain the almost Hermitian isomodel of isolagrange (or isofinsler) isospaces  $\hat{L}^n$  (or  $\hat{F}^n$ ).

We note that the natural compatibility conditions (6.6) for the isometric (6.5) and  $C\hat{\Gamma}(N)$ -connections on  $\hat{H}^{2n}$ -spaces plays an important role for developing physical models on la-isospaces. In the case of usual locally anisotropic spaces geometric constructions and d-covariant calculus are very similar to those for the Riemann and Einstein-Cartan spaces. This is exploited for formulation in a selfconsistent manner the theory of spinors on la-spaces [28], for introducing a geometric background for locally anisotropic Yang-Mills and gauge like gravitational interactions [27] and for extending the theory of stochastic processes and diffusion to the case of locally anisotropic spaces and interactions on such spaces [32]. In a similar manner we can introduce N-lifts to v- and t-isobundles in order to investigate isotopic gravitational la-models.

## 7 Isogravity on Locally Anisotropic and Inhomogeneous Isospaces

The conventional Riemannian geometry can be generally assumed to be exactly valid for the exterior gravitational problem in vacuum where bodies can be well approximated as being massive points, thus implying the validity of conventional and calculus.

On the contrary, there have been serious doubt dating back to E. Cartan on the same exact validity of the Riemannian geometry for interior gravitational problem because the latter imply internal effects which are arbitrary nonlinear in the velocities and other variables, nonlocal integral and of general non-(first)-order Lagrangian type.

Santilli [20, 21, 22, 23] constructed his isoriemannian geometry and proposed the related isogravitation theory precisely to resolve the latter shortcoming. In fact, the isometric acquires an arbitrary functional; dependence thus being able to represent directly the locally anisotropic and inhomogeneous character of interior gravitational problems.

A remarkable aspect of the latter advances is that they were achieved



by preserving the abstract geometric axioms of the exterior gravitation. In fact, exterior and interior gravitation are unified in the above geometric approach and are merely differentiated by the selected unit, the trivial value  $I = \text{diag}(1, 1, 1, 1)$  yielding the conventional gravitation in vacuum while more general realization of the unit yield interior conditions under the same abstract axioms (see ref. [10] for an independent study).

A number of applications of the isogeometries for interior problems have already been identified, such as (see ref. [22] for an outline): the representation of the local variation of the speed of light within physical media such as atmospheres or chromospheres; the representation of the large difference between cosmological redshift between certain quasars and their associated galaxies when physically connected according to spectroscopic evidence; the initiation of the study of the origin of the gravitation via its identification with the field originating the mass of elementary constituents.

As we have shown [29, 30] the low energy limits of string and superstring theories give also rise to models of (super)field interactions with locally anisotropic and even higher order anisotropic interactions. The N-connection field can be treated as a corresponding nonlinear gauge field managing the dynamics of "step by step" splitting (reduction) of higher dimensional spaces to lower dimensional ones. Such (super)string induced (super)gravitational models have a generic local anisotropy and, in consequence, a more sophisticated form of field equations and conservation laws and of corresponding their stochastic and quantum modifications. Perhaps similar considerations are in right for isotopic versions of string theories. That it is why we are interested in a study of models of isogravity with nonvanishing nonlinear isoconnection, distinguished isotorsion and, in general, non-isometric fields.

To begin our presentation let us consider a v-isobundle  $\hat{\xi} = (\hat{E}, \pi, \hat{M})$  provided with some compatible nonlinear isoconnection  $\hat{N}$ , d-isoconnection  $\hat{D}$  and isometric  $\hat{G}$  structures. For a locally N-adapted isoform we write

$$\hat{D}_{(\frac{\delta}{\delta u^\gamma})} \frac{\hat{\delta}}{\delta \hat{u}^\beta} = \hat{\Gamma}_{\beta\gamma}^\alpha \frac{\hat{\delta}}{\delta \hat{u}^\alpha},$$

where the d-isoconnection  $\hat{D}$  has the following coefficients:

$$\hat{\Gamma}_{jk}^i = \hat{L}_{jk}^i, \hat{\Gamma}_{ja}^i = \hat{C}_{ja}^i, \hat{\Gamma}_{aj}^i = 0, \hat{\Gamma}_{ab}^i = 0, \quad (7.1)$$



$$\widehat{\Gamma}^a_{jk} = 0, \widehat{\Gamma}^a_{jb} = 0, \widehat{\Gamma}^a_{bk} = \widehat{L}^a_{bk}, \widehat{\Gamma}^a_{bc} = \widehat{C}^a_{bc}.$$

The nonholonomy isocoefficients  $\widehat{w}^\gamma_{\alpha\beta}$  are as follows:

$$\widehat{w}^k_{ij} = 0, \widehat{w}^k_{aj} = 0, \widehat{w}^k_{ia} = 0, \widehat{w}^k_{ab} = 0, \widehat{w}^a_{ij} = \widehat{R}^a_{ij},$$

$$\widehat{w}^b_{ai} = -\frac{\widehat{\partial} \widehat{N}^b_a}{\widehat{\partial} \widehat{y}^a}, \widehat{w}^b_{ia} = \frac{\widehat{\partial} \widehat{N}^b_a}{\widehat{\partial} \widehat{y}^a}, \widehat{w}^c_{ab} = 0.$$

By straightforward calculations we can obtain respectively these components of isotorsion,  $\mathcal{T}(\widehat{\delta}_\gamma, \widehat{\delta}_\beta) = \mathcal{T}^\alpha_{\beta\gamma} \widehat{\delta}_\alpha$ , and isocurvature,  $\mathcal{R}(\widehat{\delta}_\beta, \widehat{\delta}_\gamma) \widehat{\delta}_\tau = \mathcal{R}^\alpha_{\beta\gamma\tau} \widehat{\delta}_\alpha$ , d-isotensors:

$$\mathcal{T}^i_{jk} = \widehat{T}^i_{jk}, \mathcal{T}^I_{ja} = \widehat{C}^I_{ja}, \mathcal{T}^I_{ja} = -\widehat{C}^I_{ja}, \mathcal{T}^i_{ab} = 0, \quad (7.2)$$

$$\mathcal{T}^a_{ij} = \widehat{R}^a_{ij}, \mathcal{T}^a_{ib} = -\widehat{P}^a_{bi}, \mathcal{T}^a_{bi} = \widehat{P}^a_{bi}, \mathcal{T}^a_{bc} = \widehat{S}^a_{bc}$$

and

$$\mathcal{R}^j_{i.kl} = \widehat{R}^j_{i.kl}, \mathcal{R}^j_{b.kl} = 0, \mathcal{R}^a_{j.kl} = 0, \mathcal{R}^a_{b.kl} = \widehat{R}^a_{b.kl}, \quad (7.3)$$

$$\mathcal{R}^i_{j.kd} = \widehat{P}^i_{j.kd}, \mathcal{R}^a_{b.kd} = 0, \mathcal{R}^a_{j.kd} = 0, \mathcal{R}^a_{b.kd} = \widehat{P}^a_{b.kd},$$

$$\mathcal{R}^i_{j.dk} = -\widehat{P}^i_{j.kd}, \mathcal{R}^i_{b.dk} = 0, \mathcal{R}^a_{j.dk} = 0, \mathcal{R}^h_{b.dk} = -\widehat{P}^h_{b.kd},$$

$$\mathcal{R}^i_{j.cd} = \widehat{S}^i_{j.cd}, \mathcal{R}^i_{b.cd} = 0, \mathcal{R}^a_{j.cd} = 0, \mathcal{R}^a_{b.cd} = \widehat{S}^a_{b.cd}$$

(for explicit dependencies of components of isotorsions and isocurvatures on components of d-isoconnection see formulas (5.4) and (5.7)).

The locally adapted components  $\mathcal{R}_{\alpha\beta} = \text{Ric}(D)(\widehat{\delta}_\alpha, \widehat{\delta}_\beta)$  (we point that in general on t-isobundles  $\mathcal{R}_{\alpha\beta} \neq \mathcal{R}_{\beta\alpha}$ ) of the **isoricci tensor** are as follows:

$$\mathcal{R}_{ij} = \widehat{R}^k_{i.jk}, \mathcal{R}_{ia} = -^{(2)}\widehat{P}_{ia} = -\widehat{P}^k_{i.k a} \quad (7.4)$$

$$\mathcal{R}_{ai} = \widehat{P}_{ai} = \widehat{P}^b_{a.ib}, \mathcal{R}_{ab} = \widehat{S}^c_{a.bc} = \widehat{S}_{ab}.$$

For scalar curvature,  $\overleftarrow{R} = Sc(\widehat{D}) = \widehat{G}^{\alpha\beta} \widehat{R}_{\alpha\beta}$ , we have

$$Sc(\widehat{D}) = \widehat{R} + \widehat{S}, \quad (7.5)$$

where  $\widehat{R} = \widehat{g}^{ij} \widehat{R}_{ij}$  and  $\widehat{S} = \widehat{h}^{ab} \widehat{S}_{ab}$ .



The **isoeinstein–isocartan equations with prescribed N–isoconnection** and **h(hh)–** and **v(vv)–isotorsions** on **v–isobundles** (compare with isoeinstein isoequations (2.11)) are written as

$$\hat{R}^{\alpha\beta} - \frac{1}{2}\hat{g}^{\alpha\beta}(\overleftarrow{R} + \hat{\Theta} - \lambda) = \kappa_1(\hat{t}^{\alpha\beta} - \hat{\tau}^{\alpha\beta}), \quad (7.6)$$

and

$$\hat{T}_{\beta\gamma}^{\alpha} + G_{\beta}^{\alpha}\hat{T}^{\tau}_{\gamma\tau} - G_{\gamma}^{\alpha}\hat{T}^{\tau}_{\beta\tau} = \kappa_2\hat{Q}_{\beta\gamma}^{\alpha}, \quad (7.7)$$

where  $\hat{Q}_{\beta\gamma}^{\alpha}$  spin–density of matter d–isotensors on locally anisotropic and homogeneous isospace,  $\kappa_1$  and  $\kappa_2$  are the corresponding interaction constants and  $\lambda$  is the cosmological constant,  $\hat{t}^{\alpha\beta}$  is a source isotensor and  $\hat{\tau}^{\alpha\beta}$  is the stress–energy isotensor and there is satisfied the generalized Freud isoidentity

$$\hat{G}_{\beta}^{\alpha} - \frac{1}{2}\delta_{\beta}^{\alpha}(\hat{\Theta} - \lambda) = \hat{U}_{\beta}^{\alpha} + \hat{\delta}_{\rho}\hat{V}^{\alpha\rho}_{\beta}, \quad (7.8)$$

where

$$\begin{aligned} \hat{G}_{\beta}^{\alpha} &= \hat{R}^{\alpha}_{\beta} - \frac{1}{2}\delta_{\beta}^{\alpha}\overleftarrow{R}, \\ \hat{U}_{\beta}^{\alpha} &= -\frac{1}{2}\frac{\hat{\delta}\hat{\Theta}}{\hat{\partial}(\hat{D}_{\alpha}\hat{g}^{\gamma\delta})}\hat{D}_{\beta}\hat{g}^{\gamma\delta} \end{aligned}$$

and

$$\begin{aligned} \hat{V}^{\alpha\rho}_{\beta} &= \frac{1}{2}[\hat{g}^{\gamma\delta}(\delta_{\beta}^{\alpha}\hat{\Gamma}_{\alpha\delta}^{\rho} - \delta_{\delta}^{\alpha}\hat{\Gamma}_{\alpha\beta}^{\rho}) + \\ &\hat{g}^{\rho\gamma}\hat{\Gamma}_{\beta\gamma}^{\alpha} - \hat{g}^{\alpha\gamma}\hat{\Gamma}_{\beta\gamma}^{\rho} + (\delta_{\beta}^{\rho}\hat{g}^{\alpha\gamma} - \delta_{\beta}^{\alpha}\hat{g}^{\rho\gamma})\hat{\Gamma}_{\gamma\rho}^{\rho}]. \end{aligned}$$

By using decompositions (7.1)–(7.5) it is possible an explicit projection of equations (7.6)–(7.8) into vertical and horizontal isocomponents (for simplicity we omit such formulas in this work).

Equations (7.6) constitute the fundamental field equations of Santilli isogravitation [20, 21, 22, 23] written in this case for vector isobundles provided with compatible N- and d–isoconnection and isometric structures. The algebraic equations (7.7) have been here added, apparently for the first time for isogravity with isotorsion (see also [27, 33, 31] for locally anisotropic gravity and supergravity) in order to close the system of gravitational isofield equations (really we have also to take into account the system of constraints



(4.5) if locally anisotropic inhomogeneous gravitational isofield is associated to a d-isometric (4.6), or to a d-isometric (6.5) if the isogravity is modelled on a tangent isobundle). It should be noted here that the system of isogravitational field equations (7.8) presents a synthesis for vector isobundles of equations introduced by Anastasiei, [1] and [14], and of equations (2.11) and (2.12) considered in the Santilli isotheory.

We note that on la-isospaces the divergence

$$D_\alpha[\hat{G}^\alpha_\beta - \frac{1}{2}\delta^\alpha_\beta(\hat{\Theta} - \lambda)] = \hat{U}_\beta \quad (7.9)$$

does not vanish (this is a consequence of generalized isobianchi (5.8), or (5.9), and isor Ricci isoidentities (5.11), or (5.12)). The problem of nonvanishing of such divergences for gravitational models on vector bundles provided with nonlinear connection structures was analyzed in [1] and [14].

The problem of total conservation laws on isospaces has been studied in detail in ref. [20] by reformulating all isospaces considered in that paper in terms of the isominkowskian space, with consequential elimination of curvature which permits the construction of a universal symmetry and related total conservation laws for all possible isometric.

The latter studies concerning vector isobundles with N-isoconnections will be considered in some future works.

We end this subsection by emphasizing that isofield equations of type (7.6)–(7.8) can be similarly introduced for the particular cases of locally anisotropic isospaces with metric (6.5) on  $T\tilde{M}$  with coefficients parametrized as for the isolagrange, (6.3), or isofinsler, (6.4), isospaces.

## 8 Concluding Remarks and Further Possibilities

One of the most important aspects we attempted to convey in this work is the possibility to formulate isotopic variants of extended Finsler geometry and the application of this isogeometric background in contemporary theoretical and mathematical physics. The approach adopted here provides us an essentially self-contained, concise and significantly simple treatment of the material on bundle isospaces enabled with compatible isotopic nonlinear and distinguished isoconnections and isometric structures.



A remarkable features worth recalling is that the considerable broadening of the capabilities of the isothory via the additional of nonlinear, nonlocal and noncanonical effects, is done via the same abstract axioms of the conventional formulations.

In this paper we have discussed the basic geometric constructions for isotopic spaces with inhomogeneity and local anisotropy. We have computed the distinguished isotorsions and isocurvatures. It was shown how to write a manifestly isotopic model of gravity with locally anisotropic and inhomogeneous interactions of isofields. The assumptions made in deriving the results are similar to those for the geometry of isomanifolds and to the isofield theory.

There are various possible developments of the ideas presented here. One of the necessary steps is the definition of locally anisotropic and inhomogeneous isotopic spinors and explicit constructions of physical models with isospinor, isogauge and isogravitational interactions on locally anisotropic isospaces. The problem of formulation of conservation laws on locally anisotropic and inhomogeneous isospaces and for locally anisotropic and inhomogeneous isofield interactions presents a substantial interest for investigations. Here we add the theory of isostochastic processes, the supersymmetric extension of the concept of isothory as well possible generalizations of the mentioned constructions for higher order anisotropies in string theories. These tasks remain for future research.

## Acknowledgement

The author wish to express generic thanks to the referees for a detailed control and numerous constructive suggestions.



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